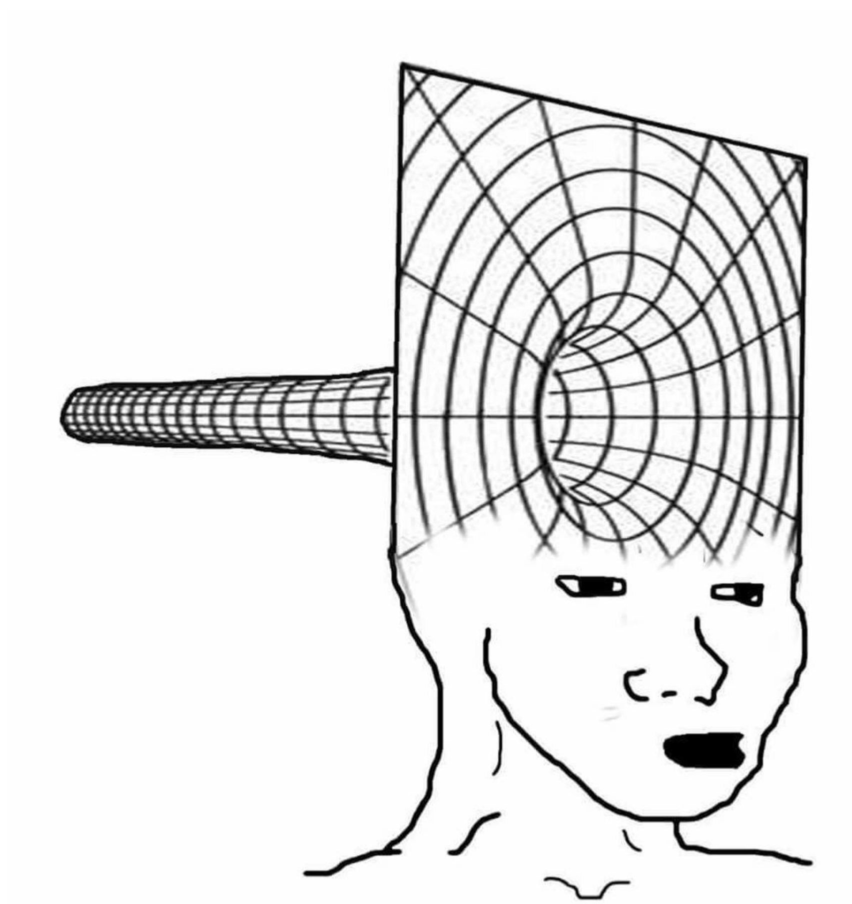
Categories

Daisuke Natthaworn Sakai

September 6, 2020



Useful links:
https://www.youtube.com/playlist?list=PLCTMeyjMKRkoS699U0OJ3ymr3r01sI08l
https://webusers.imj-prg.fr/~pierre.schapira/lectnotes/AlTo.pdf
https://ncatlab.org/nlab/show/small+category
https://bartoszmilewski.com/2016/04/18/adjunctions/
https://ncatlab.org/nlab/show/limits+commute+with+limits
https://ncatlab.org/nlab/show/adjoint+functor#GeneralAdjunctsInTermsOfAdjunctionUnitC
https://www.youtube.com/watch?v=eXeJx0hByeY
https://ncatlab.org/nlab/show/limit#global_definition_in_terms_of_adjoint_of_the_const
https://ncatlab.org/nlab/show/pullback

Contents

These are a set of notes on category theory for ease of reference.

Indeed, there are a plethora of attempts to put category theory on a axiomatic logical foundation. In any case, the theory of sets still presents itself as the most useful and didactic and resisted the tests of time, and hence is adopted as convention by virtually all mathematicians. This text assumes ZFC, and we assume Grothendieck's axiom, as these suffice to prove the useful results in other fields of mathematics. As always, the theory presented is entirely self contained. We only assume knowledge presented in "Part I: Prerequisites" in the text Foundations of Analysis (FOA), also written by the current author.

Corrections to mistakes and suggestions can be emailed to dnsakai1729@gmail.com

Remark. For logical completeness, when I say that something is equal, I mean that it is equal, not isomorphic.

1 Categories

1.1 Definitions

Definition 1. $\mathbb{A} = (Ob(\mathbb{A}), \mathrm{Hom}(\mathbb{A}), \mathrm{Hom}, \circ)$ is a "category" iff:

1. $\mathrm{Hom} : Ob(\mathbb{A}) \times Ob(\mathbb{A}) \to \mathrm{Hom}(\mathbb{A})$ is a map such that:

   (a) Hom is a cover of $Hom(\mathbb{A})$

   (b) The image of any two elements in the image of Hom are pairwise disjoint. Elements in $Hom(\mathbb{A})$ are called "morphisms".

2. $\circ$ maps from $Ob(\mathbb{A}) \times Ob(\mathbb{A}) \times Ob(\mathbb{A})$ to $\mathcal{P}((\mathrm{Hom}(\mathbb{A}) \times \mathrm{Hom}(\mathbb{A})) \times \mathrm{Hom}(\mathbb{A}))$, where:

   (a) $A, B, C \in Ob(\mathbb{A})$, we have the map

   $$\circ_{(A,B,C)} : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

   In this case, we will simply write "$\circ$" instead of "$\circ_{(A,B,C)}$" by abuse of notation, and for $\alpha \in \mathrm{Hom}(A, B)$, and $\beta \in Hom(B, C)$, denote $\circ(\alpha, \beta) := \alpha \circ \beta$.

   (b) Given (a) hence, for all $A \in Ob(\mathbb{A})$ there exists an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that for all $B \in Ob(\mathbb{A})$, we have $\forall f \in \mathrm{Hom}(A, B) : 1_A \circ f = f$ and $\forall f \in \mathrm{Hom}(A, B) : f \circ 1_A = f$. It is immediately verified that the identity morphism is unique. (If we assume that in condition 1, Hom is a partition.)

   (c) For all $A, B, C, D \in Ob(\mathbb{A})$, and for all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$, we have $(f \circ g) \circ h = f \circ (g \circ h)$. (Note that if (a) holds, then both sides of the equality must exist; that is, we have that both $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are in $\mathrm{Hom}(A, D)$, and are equal; the equality is in fact always, "defined")

The condition 1. (b) can be ommitted. When we are given a category with this weaker condition, we can always associate the objects with the original morphisms to create an identical looking category satisfying 1. (b) in the following way. Suppose $\overline{\mathrm{Hom}}$ is a surjection. Define $\mathrm{Hom}(A, B) := \{(A, f, B) \mid f \in \overline{\mathrm{Hom}}(A, B)\}$. If $\overline{\circ}$ is the original composition, define a new composition by $(A, f, B) \circ (B, g, D) := (A, g \overline{\circ} f, D)$. It is routine verification to obtain that the new composition then satisfies condition 2. When we discuss categories we will always implicitely assume this construction hence, but will not explicitely mention it, and will write f for (A, f, B), and $g \circ f$ for $(A, g \circ f, D)$, by abuse of notation.

Although we have given the definition of a category in completely set-theoretic terms, more often than not, the working mathematician does not have time to use such a formal definition. Instead it is best to use the following informal rephrasing of the definition (with the weaker condition):

Definition 2. $\mathbb{A}$ is a "category" iff:

1. Given any two objects A, B of $\mathbb{A}$ we associate a set $\mathrm{Hom}(A, B)$. For objects A, B, we write, as a logical sentence, "$f : A \to B$" iff $f \in \mathrm{Hom}(A, B)$. In this case, f is called a morphism. Then if $f : A \to B$ and $g : B \to C$ then the composition is defined and $g \circ f : A \to C$.

2. When $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

3. For $A \in Ob(\mathbb{A})$ there exists an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that we have $1_A \circ f = f$ and $f \circ 1_A = f$, whenever the composition is defined.

Remark. Note that given two objects X and Y, it is not necessary that the collection $\mathrm{Hom}(X, Y)$ need not be non-empty, but $\mathrm{Hom}(X, X)$ is always non-empty, due to the identity morphism.

When verifying that a mathematical structure is a category, the less formal definition is more suitable for use. However in certain cases, one needs to use the more rigorous definition, which arises, for example, when verifying that a category is equal to another category.

The notation $\mathrm{Hom}(A, B)$ makes reference to the word "homomorphism" even though in general, the category at hand might not be one that has homomorphisms as its morphisms. One may also write $\mathrm{Mor}(A, B)$ in deference to the word "morphism" (In particular, Lang's Algebra uses this notation) but we shall stick with the notation that is standard in modern literature.

Example 3. The category of relations is denoted **Rel**. The category of sets is denoted **Set**. The category of (left) A-modules is denoted **Mod(A)**. The category of all topological spaces is denoted **Top**. The category of all rings is denoted **Ring**. The cateogry of all groups is denoted **Grp**.

The empty category, which has no objects and hence no morphisms is denoted **Empty**. It can also be denoted as $\varnothing$, although this is not advisable, because the empty category is not identical to the empty set. The terminal category is the category that has exactly one object, and one morphism: the identity from the object to itself. The terminal category is often denoted as $\{\bullet\}$ or as pt.

Given a partially ordered set $(I, \leq)$, we are able to define its corresponding partial order category, which we shall denote here as $\mathbf{Poset}(I, \leq)$, which takes the objects as elements of I, and $\mathrm{Hom}(i, j)$ to contain exactly one element, the empty set, iff $i \leq j$, and exactly no elements otherwise. Define the composition $\varnothing \circ \varnothing = \varnothing$. It is easily confirmed that this gives a category.

We are able to similarly define the category associated with a pre-ordered set $\mathbf{Proset}(I, \leq)$.

Definition 4. Given categories C, S, we shall say that S is a "subcategory" of C iff:

1. $Ob(S) \subset Ob(C)$
2. $\forall X, Y \in Ob(S) : \mathrm{Hom}_S(X, Y) \subset \mathrm{Hom}_C(X, Y)$
3. If $f \in \mathrm{Hom}_S(X, Y)$ and $g \in \mathrm{Hom}_S(Y, Z)$, then $g \circ_S f = g \circ_C f$
4. If $id_X \in \mathrm{Hom}_C(X, X)$ is the identity morphism, then $id_X \in \mathrm{Hom}_S(X, X)$

Any category is a subcategory of itself. The empty category is a subcategory of any category.

A category is called "discrete" iff all its morphisms are identity morphisms. Given arbitrary set I, we are able to define the discrete category of I, which we shall denote $\mathbf{Disc}(I)$, where the objects are exactly the elements of I.

1.2 Universes

In algebraic geometry, algebraic topology, and virtually any other sensible area of mathematics, it suffices to understand categories within the framework of Universes. Grothendieck Universes allows us to construct a copy of ZFC in ZFC itself without problems concerning ourselves with paradoxes like the "set of all sets". Elements of a Grothendieck Universe $\mathbb{U}$ are "sets" which we can study, but not subsets of $\mathbb{U}$.

Definition 5. A set $\mathcal{U}$ is called a "Grothendieck Universe" iff it satisfies the following properties:

1. $x \in \mathcal{U}$ and $y \in x$ implies that $y \in \mathcal{U}$
2. $x \in \mathcal{U}$ implies $\mathcal{P}(x) \in \mathcal{U}$
3. For all $I \in \mathcal{U}$ and functions $u : I \to \mathcal{U}$, the union of $Im(u)$ is in $\mathcal{U}$.
4. $\varnothing \in \mathcal{U}$

We take as an axiom, that for any set S, there exists a Grothendieck Universe $\mathcal{U}$ such that $S \in \mathcal{U}$.

Proposition 6. *Any Grothendieck Universe $\mathcal{U}$ is a model of ZFC.*

We omit the proof of the above proposition.

A set is called "$\mathcal{U}$-small" iff it is in $\mathcal{U}$. A set is called "$\mathcal{U}$-large" iff it is not in $\mathcal{U}$. A set is called "$\mathcal{U}$-moderate" iff it is a subset of $\mathcal{U}$. A set is called "essentially $\mathcal{U}$-small" iff it is set isomorphic to some $\mathcal{U}$-small set. The same definition applies for essentially $\mathcal{U}$-large and essentially $\mathcal{U}$-moderate sets.

A category $\mathbb{A}$ is called $\mathcal{U}$-small iff $Ob(\mathbb{A})$ and $Hom(\mathbb{A})$ are $\mathcal{U}$-small. $\mathbb{A}$ is called "structurally $\mathcal{U}$-small" iff $Ob(\mathbb{A})$ and $Hom(\mathbb{A})$ are essentially $\mathcal{U}$-small (for which it is sufficient to say that $Hom(\mathbb{A})$ is essentially $\mathcal{U}$-small). $\mathbb{A}$ is called "locally $\mathcal{U}$-small" iff each of its hom-sets is $\mathcal{U}$-small.

When it is clear that we are talking about a fixed universe $\mathcal{U}$, we can omit it from our notation.

When we talk about sets, we will use the word "set" to mean any set in ZFC+U. In particular, to be explicit, when we say that "A is an arbitrary set", we mean that A is not necessarily $\mathcal{U}$-small.

Example 7. The categories **Rel**, **Set**, **Mod(A)**, **Top**, **Ring**, **Grp**, **Ab** are locally small categories, but not small.

As in the above example, in essentially all cases, we shall be dealing with locally small categories, and this is the definition that is most important to remember.

Take some universe $\mathcal{U}$. Henceforth, when we say that something is small, we mean that it is $\mathcal{U}$-small. We write **Set** to denote the category of all $\mathcal{U}$-small sets, as it is not possible to discuss the category of all sets, but only possible to discuss the category of all sets that are small with respect to some universe.

1.3 Opposite and Product Category

Definition 8. Given a category $\mathbb{A}$, the opposite category, denoted $\mathbb{A}^{\text{op}}$, is the unique category such that:

1. $Ob(\mathbb{A}) = Ob(\mathbb{A}^{\text{op}})$, and $Hom(\mathbb{A}) = Hom(\mathbb{A}^{\text{op}})$
2. Given any two objects $X, Y \in Ob(\mathbb{A})$, we have that $Hom_{\mathbb{A}}(X,Y) = Hom_{\mathbb{A}^{\text{op}}}(Y,X)$
3. Given any two composible morphisms $\alpha, \beta \in Hom(\mathbb{A}^{\text{op}})$, we have that $\beta \circ_{\text{op}} \alpha = \alpha \circ \beta$

It is easily verified that the category that satisfies the above three conditions is unique, and it is obvious that the opposite category does indeed exist by simply defining it. We notice that $(\mathbb{A}^{\text{op}})^{\text{op}} = \mathbb{A}$.

There is a philosophical point to be noted here. The way in which we symbolically chose to represent a category is entirely aribtrary in the sense that there is no reason that an arrow must be pointing one way instead of the other. That is, a map from A to B can be equally well represented by writing $A \leftarrow B$. Therefore we see that there is something essentially extraneous in our theory; $\mathbb{A}$ and $\mathbb{A}^{\text{op}}$ in essense describe the exact same ideas.

The point that it does not matter which way we define the direction of an arrow is of importance when dealing with dual notions.

There is a formal way to define what is meant by the "dual notion" of a concept in category theory, and requires notions of mathematical logic and formal languages. Without going too deeply in this realm (as the point of these notes is to develop category theory for use in other fields, and not the study of category theory itself), it would be sufficient to keep the following informal definition in mind (from wikipedia):

Suppose P is some logical sentence (definition, proposition, etc) regarding a category C (that is, anything mentioned in this sentence must be an object, morphism, etc) of this category. Then P^{op} is the dual sentence iff swapping the words "source" and "target" and swapping the "$g \circ f$" with "$f \circ g$" in P obtains P^{op}. In this text, when we use the word "dual" we shall only use it in an informal context.

One may use the expression "formally dual" to say that a sentence is dual.

Definition 9. In general we have the notion of the product of an arbitrary indexed set of categories. However, in practical purposes, such as in algebraic topology, we will only need to reason with the product of two categories. It is, however more useful to consider the product of two categories as the product of the two indexed categories which will allow us to simplify proofs.

Given two categories $\mathbb{A}$ and $\mathbb{B}$, put

$$Ob(\mathbb{A} \times \mathbb{B}) := Ob(\mathbb{A}) \times Ob(\mathbb{B})$$

$$Hom(\mathbb{A} \times \mathbb{B}) := Hom(\mathbb{A}) \times Hom(\mathbb{B})$$

Given any two pairs of objects $(X, Y), (X', Y') \in Ob(\mathbb{A} \times \mathbb{B})$, we associate the set

$$Hom((X, Y), (X', Y')) := Hom(X, Y) \times Hom(X', Y')$$

Given two morphisms $(\alpha, \beta) : (X, Y) \to (X', Y')$, $(\alpha', \beta') : (X', Y') \to (X'', Y'')$, define their composition as

$$(\alpha, \beta) \circ (\alpha', \beta') := (\alpha \circ \alpha', \beta \circ \beta')$$

Then it is easily verified that $\mathbb{A} \times \mathbb{B}$ is a category.

In general, given $\{\mathbb{A}_i\}_i$, put

$$Ob(\prod \mathbb{A}_i) := \prod Ob(\mathbb{A}_i)$$

$$\mathrm{Hom}(\prod \mathbb{A}_i) := \prod \mathrm{Hom}(\mathbb{A}_i)$$

Given any two indexed set of objects $\{X_i\}_i, \{Y_i\}_i \in Ob(\prod \mathbb{A}_i)$:, we associate the set

$$\mathrm{Hom}(\{X_i\}_i, \{Y_i\}_i) := \prod \mathrm{Hom}_{\mathbb{A}_i}(X_i, Y_i)$$

Given two morphisms $\{\alpha_i\}_i : \{X_i\}_i \to \{Y_i\}_i$, $\{\beta_{i\}_i} : \{Y_i\}_i \to \{Z_i\}_i$, define their composition as

$$\{\beta\}_i \circ \{\alpha_i\}_i := \{\beta_i \circ \alpha_i\}_i$$

Then it is easily verified that this obtains a category.

1.4 Definitions Regarding Morphisms

Given element $f \in Hom(\mathbb{A})$, we call f a "morphism" or an "arrow". As mentioned in the above definition, we will, as a logical statement, write $f : A \to B$ (or $A \xrightarrow{f} B$) iff $f \in Hom(A, B)$. Then it is easy to see that whenever $A \xrightarrow{f} B \xrightarrow{g} C$, the composition $g \circ f$ exists in the category.

We say that a morphism f is an identity whenever it is an identity element for some object X. By the definition of a category, we see that this element is unique.

When $f : A \to B$ and $g : B \to C$, we say that $f \circ g$ is the "composition" of f and g.

Given morphism $f : A \to B$, we say that A is the "source" of f, and B is the "target" of f.

An element in $Ob(\mathbb{A})$ is called an "object of $\mathbb{A}$," and an element in $Mor(\mathbb{A})$ is called a "morphism of $\mathbb{A}$," or an "arrow of $\mathbb{A}$". The image of an element under Mor is said to be a "hom-set" (which may also written as homset).

A morphism that is right-cancellable is called a "monomorphism", and a morphism that is left-cancellable is called an "epimorphism". Morphisms are called "monic" whenever they are monomorphisms, and "epic" whenever they are epimorphisms. We easily see that a monomorphism in C is an epimorphism in C^{op} and conversely, and that an epimorphism in C is a monomorphism in C^{op} and conversely.

For morphism $f \in \text{Hom}(A, B)$, when $g \in \text{Hom}(B, A)$ and $f \circ g = id_A$ and $g \circ f = id_B$, we say that "g is an inverse of f." If an inverse of f exists, then f is said to be an "isomorphism". A morphism that is an isomorphism is also called "invertible". An inverse is immediately verified to be unique. The following statement holds:

Proposition 10. *A morphism f is an identity iff there exists morphism g such that $f \circ g$ and $g \circ f$ are identity morphisms.*

We say that "A is isomorphic to B" when such a function f exists. Clearly, if A is isomorphic to B, then B is isomorphic to A, so we simply say that "A and B are isomorphic."

Definition 11. A category C is called "balanced" iff:

for all morphisms $f \in \text{Hom}(C)$, if f is both monic and epic, then f is an isomorphism.

Proposition 12. *We note the following:*

1. *The identity morphism is an isomorphism*

2. *The compositions of two isomorphisms is an isomorphism*

3. *The inverse of an isomorphism is an isomorphism*

1.5 Terminal and Initial Objects (Universal Objects)

For an object A in a category:

- A is called "universally repelling" (or "initial") iff: for all objects B the collection $\mathrm{Hom}(A, B)$ is a singleton

- A is called "universally attracting" (or "terminal" or "final") iff: for all objects B the collection $\mathrm{Hom}(B, A)$ is a singleton

- A is called "zero" iff it is both inital and terminal

It is to be noted here that the usage of the word "universal" has no connection to the Grothendieck Universe.

We note the following facts regarding universal objects.

- Universally repelling or attracting objects are unique under isomorphism.

- If A is terminal in C, then it is initial in C^{op} and conversely

- If A is initial in C, then it is terminal in C^{op} and conversely

Proposition 13. *If $F : C \to D$ is an isomorphism of categories, then:*

1. *If X is terminal in C, then $F(X)$ is terminal in D (and conversely)*

2. *If X is initial in C, then $F(X)$ is initial in D (and conversely)*

Proof. After proving 1, replace C with C^{op}, and D with D^{op} to obtain 2. $\square$

Corollary. *If C and D are isomorphic categories, then:*

1. *If Chas a terminal object, then D has a terminal object*

2. *If Chas an initial object, then D has an initial object*

Remark. It is to be noted here that although it may seem obvious or trivial, this proposition allows us to fill in technicalities in proofs that otherwise would not be complete. In particular as we will see later, although the equalizer is not quite exactly a limit, the equalizer category is isomorphic to the limit category of some functor. Then, when a category admits limits, it automatically admits equalizers.

We denote zero objects by the symbol "0". Such objects may or may not exist depending on the category at hand.

Given a category $\mathbb{A}$, and object A of $\mathbb{A}$, we denote the set $\mathrm{End}(A) := \mathrm{Hom}(A, A)$. We call this set the set of all endormorphisms of A, and an element in the set is called an "endomorphism of A". Phrased differently, an arrow f is called an "endomorphism," iff there exists object A such that $f \in \mathrm{Hom}(A, A)$. An endomorphism that is also an isomorphism is called an "automorphism." We denote the set $\mathrm{Aut}(A) := \{f \in \mathrm{End}(A) \mid f \text{ is an isomorphism}\}$, as the set of all automorphisms of A.

When discussing multiple categories at once, for a category $\mathbb{A}$, we denote $\mathrm{Hom}_{\mathbb{A}}(A, B)$, $\mathrm{End}_{\mathbb{A}}(A)$, $\mathrm{Aut}_{\mathbb{A}}(A)$ (and so on) to respectively denote $\mathrm{Hom}(A, B)$, $\mathrm{End}(A)$, $\mathrm{Aut}(A)$ (and so on) to in the category $\mathbb{A}$.

1.6 Constant and Coconstant morphisms

Definition 14. Suppose C is a category. A morphism $f : A \to B$ is called "constant" iff for all morphisms $g, h : X \to A$ we have the equality $f \circ g = f \circ h$. A constant morphism is also called a "left zero morphism".

A morphism $f : A \to B$ is called "coconstant" iff it is constant when considered in the opposite category. That is to say that for all morphisms $g, h : B \to X$ in C, we have the equality $g \circ f = h \circ f$ in C. A coconstant morphism is also called a "right zero morphism". We will say that a morphism is a "zero morphism" iff it is both right and left zero.

It is useful to remember that:

1. If f is constant, then $f \circ g = f \circ id = f$
2. If f is coconstant, then $g \circ f = id \circ f = f$

We also observe that if a morphism f is constant in C, then it is coconstant in C^{op} and conversely, and (hence) if f is coconstant in C, then it is constant in C.

Proposition 15. *If C is a category with a zero object, then:*

1. *A morphism to 0 is constant*
2. *A morphism from 0 is coconstant*
3. *A morphism factors through a zero object iff it is a zero morphism.*

Proof. We prove the statements one by one.

1. Immediate from definition of terminal object.
2. Immediate from definition of initial object.
3. If $f : X \to Y$ factors through 0, then put $f = 0_{0,Y} \circ 0_{X,0}$. Then we have equalities $(\alpha \circ 0_{0,Y}) \circ 0_{X,0} = (\beta \circ 0_{0,Y}) \circ 0_{X,0}$ and $0_{0,Y} \circ (0_{X,0} \circ \alpha) = 0_{0,Y} \circ (0_{X,0} \circ \beta)$, so f is a zero morphism. Conversely, suppose $f : X \to Y$ is a zero morphism. Due to the definition of a zero object, we have the diagram

$$\begin{tikzcd} & 0 \arrow[dl, "\beta_X"'] \arrow[dr, "\beta_Y"] & \\ X \arrow[rr, "f"] \arrow[dr, "\alpha_X"'] & & Y \arrow[dl, "\alpha_Y"] \\ & 0 & \end{tikzcd}$$

   that commutes. We have, then, that $\alpha_X = \alpha_Y \circ f$, and $\beta_Y = f \circ \beta_X$. Therefore $\beta_Y \circ \alpha_X = f \circ \beta_X \circ \alpha_Y \circ f$. We have that $\beta_X \circ \alpha_Y \circ f$ and id_X are both maps from X to X and since f is constant, we obtain $\beta_Y \circ \alpha_X = f \circ id_X$.

$\square$

Example 16. In the category **Sets**, we see that constant functions are constant morphisms. Conversely, constant morphism are constant functions.

Proof. If $f : A \to B$ is a constant morphism, we have that $f \circ id_A = f \circ x$ where x denotes the constant function mapping all elements to $x \in A$. $\square$

Example 17. In the category $\mathbf{Mod}(A)$, we see that zero maps are coconstant morphisms, and conversely, coconstant morphisms are zero maps.

Proof. If $f : A \to B$ is coconstant, then $id_B \circ f = 0 \circ f$, where 0 is the zero homomorphism. $\square$

Definition 18. Given category C, suppose we are given a map $0 : Ob(C) \times Ob(C) \to \mathrm{Hom}(C)$ that we shall call a "null map", that satisfies $0(X,Y) \in \mathrm{Hom}(X,Y)$ such that $0_{X,Y} := 0(X,Y)$ is some morphism in C (that could be but is not necessarily a zero morphism in C). Further, suppose that this map satisfies the following condition:

- Given any $X, Y, Z \in Ob(C)$, and any morphisms $g : X \to Y$, $f : Y \to Z$ in C, the diagram

$$\begin{tikzcd} X \arrow[r, "0_{X,Y}"] \arrow[d, "f"'] \arrow[dr, "0_{X,Z}"] & Y \arrow[d, "g"] \\ Y \arrow[r, "0_{Y,Z}"'] & Z \end{tikzcd}$$

  commutes. That is to say that appending a morphism in before or after a zero morphism makes it a zero morphism.

We shall say that C is a "null-map category" iff such a map 0 exists.

When 0 is a null-map on category C we can define $0^{\mathrm{op}}(X,Y) := 0(Y,X)$. Then it is easily seen that this is a null map on C^{op}. Therefore if C is a null-map category, then C^{op} is a null-map category.

Proposition 19. *Given any category C, if a null-map exists, then it is unique.*

Proof. Suppose we have two maps 0 and $0'$ which define two categories of zero morphisms of C.

Suppose we have $0_{X,Y}, 0'_{X,Y} : X \to Y$. Then we have the commutativity of

$$\begin{tikzcd} X \arrow[r, "0_{X,Y}"] \arrow[d, "0'_{X,Y}"'] \arrow[dr, "0_{X,Y}"] & Y \arrow[d, "0'_{Y,Y}"] \\ Y \arrow[r, "0_{Y,Y}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "0'_{X,Y}"] \arrow[d, "0_{X,Y}"'] \arrow[dr, "0'_{X,Y}"] & Y \arrow[d, "0_{Y,Y}"] \\ Y \arrow[r, "0'_{Y,Y}"'] & Y \end{tikzcd}$$

which shows that $0_{X,Y} = 0'_{X,Y}$. $\square$

Any category C that has a zero object, we are able to associate a zero morphism for each pair of objects.

Proof. Given X, Y, define $0_{X,Y}$ as the composition of

$$X \longrightarrow 0 \longrightarrow Y$$

This gives a zero morphism for each pair of objects which defines a null-map. $\square$

In particular suppose that C has a zero morphism for each pair of objects (X, Y) of C. Then define $0 : Ob(C) \times Ob(C) \to \mathrm{Hom}(C)$ that chooses the zero morphism of (X, Y). We see that the conditions are satisfied to call C a "null map category".

We see that zero morphisms are generalizations of the notion of homomorphisms which map to a zero element in some algebraic structure. For example, a group homomorphism $f : G \to H$ by the association $f : x \mapsto 0_H$ for all $x \in G$ is a zero morphism. First we see that zero objects allows us to treat the general notion of "maps which bring all elements to zero" in an aribtrary category, while zero morphisms allows us to treat these maps without mentioning zero objects. We get a different way of encapsulating the notion by considering null-map categories, which allows us to treat these maps without mentioning zero morphisms.

1.7 Underlying Category

When C and D are categories (satsifying either the weaker or stronger definition of a category), we shall say that "C is an underlying category of D" iff given any composible two morphisms $\alpha, \beta \in \mathrm{Hom}(D)$ in D, they are also composible in C, and the morphism $\beta \circ_D \alpha$ is exactly the same element as $\beta \circ_C \alpha$.

Example 20. **Set** is the underlying category of **Grp**, **Top**, **Mod**(A), etc. in the weaker definition. It is not their underlying category in the stronger definition.

Proposition 21. *Suppose C is a category (with the weak definition) with* **Set** *(with its weak definition) as an underlying category. Then in C, the identity of any object X is the identity function on X.*

Proof. Suppose C is a category via the weak definition. Then if β is a monomorphism in C, then $\beta \circ_C \alpha = \beta \circ_C \alpha$ $\square$

Proposition 22. *Suppose C is a category (with either the weak or strong definition) with* **Set** *(with its weak definition) as an underlying category. Then in C, every monomorphism is injective, and every epimorphism is surjective.*

Proof. Suppose C is a category via the weak definition. Then if β is a monomorphism in C, $\beta \circ_C \alpha = \beta \circ_C \alpha$ $\square$

1.8 Jargon

Definition 23. "Factor Through" abc

2 Functors

2.1 Functors

Given two categories $\mathbb{A}$ and $\mathbb{B}$, a covariant functor from $\mathbb{A}$ to $\mathbb{B}$ is an ordered pair of maps $(F, \overline{F})$, such that $F : Ob(\mathbb{A}) \to Ob(\mathbb{B})$ and $\overline{F} : \mathrm{Hom}(\mathbb{A}) \to \mathrm{Hom}(\mathbb{B})$, such that:

1. If $f : X \to Y$ in $\mathbb{A}$, then $\overline{F}(f) : F(X) \to F(Y)$ in $\mathbb{B}$
2. For identity $id_X : X \to X$ in $\mathbb{A}$, we have that $\overline{F}(id_X)$ is identity in $\mathbb{B}$
3. Given any two composible morphisms $\alpha, \beta \in Mor(\mathbb{A})$, we have that $\overline{F}(\beta \circ_{\mathbb{A}} \alpha) = \overline{F}(\beta) \circ_{\mathbb{B}} \overline{F}(\alpha)$

We immediately observe that $\overline{F}(id_X) = id_{F(X)}$.

The dual notion of a covariant functor is the contravariant functor. Given two categories $\mathbb{A}$ and $\mathbb{B}$, a contravariant functor from $\mathbb{A}$ to $\mathbb{B}$ is an ordered pair of maps $(F, \overline{F})$, such that $F : Ob(\mathbb{A}) \to Ob(\mathbb{B})$ and $\overline{F} : Hom(\mathbb{A}) \to Hom(\mathbb{B})$, such that:

1. If $f : X \to Y$ in $\mathbb{A}$, then $\overline{F}(f) : F(Y) \to F(X)$ in $\mathbb{B}$
2. For identity $id_X : X \to X$ in $\mathbb{A}$, we have that $\overline{F}(id_X) : F(X) \to F(X)$ is identity in $\mathbb{B}$
3. Given any two composible morphisms $\alpha, \beta \in Hom(\mathbb{A})$, we have that $\overline{F}(\beta \circ_{\mathbb{A}} \alpha) = \overline{F}(\alpha) \circ_{\mathbb{B}} \overline{F}(\beta)$

We observe:

1. Functors preserve isomorphisms

When $(F, \overline{F})$ is a functor from C to D, we say that it is:

1. Full iff the restriction map $\overline{F} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is surjective for all objects A, B
2. Faithful iff the restriction map $\overline{F} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is injective for all objects A, B
3. Fully faithful iff $\overline{F}$ is bijective
4. Conservative iff $\overline{F}$ reflects isomorphisms, that is, whenever $\overline{F}(\alpha)$ is an isomorphism in D, α is an isomorphism in D.
5. Essentially surjective iff for all $Y \in Ob(D)$ there exists $X \in Ob(C)$ such that $F(X)$ is isomorphic to Y in D.

We may abbreviate "fully faithful" as "f.f." or "ff". It is easily verified that a fully faithful functor is necessarily conservative.

We note that there exists exactly one functor from the empty category to any category C. One also notes that there exists exactly one functor from any category C to the terminal category.

Given categories C and D, and object $A \in Ob(C)$, we shall say that the functor $c_A : D \to C$ is constant on A iff it associates

$$X \mapsto A$$

$$f \mapsto id_A$$

for all objects X and morphisms f.

2.2 Fullness and Faithfulness

Proposition 24. *If functor $(F, \overline{F})$ is faithful, then:*

1. *It reflects monomorphisms and epimorphisms*

Proof. Easy. □

Proposition 25. *If functor $(F, \overline{F})$ is fully faithful, then:*

1. *It is conservative.*

Proof. Suppose that $(F, \overline{F})$ is a functor from C to D.

1. Suppose α is a morphism in C and $\overline{F}(\alpha)$ is an isomorphism. Take morphism t in D such that $t \circ \overline{F}(\alpha)$ and $\overline{F}(\alpha) \circ t$ are identity. Take morphism β such that $\overline{F}(\beta) = t$, we obtain that $\overline{F}(\beta \circ \alpha)$ and $\overline{F}(\alpha \circ \beta)$ are identities. By faithfulness, we have that $\beta \circ \alpha$ and $\alpha \circ \beta$ are identities.

□

Example 26. The inclusion functor $(F, \overline{F})$ from **Ab** to **Grp** is one that is fully faithful such that F is not injective.

The forgetful functor from **Grp** to **Set** is a faithful functor that is not full.

2.3 Composition of Functors

Given any two functors $(F, \overline{F})$ and $(G, \overline{G})$, if the respective functions are composible, then define the composition of two functors by $(G, \overline{G}) \circ (F, \overline{F}) := (G \circ F, \overline{G} \circ \overline{F})$. Then the composition of functors is a functor. Given category C, we shall denote $id_C = (id_{Ob(C)}, id_{\mathrm{Hom}(C)})$ as the identity functor.

It is canonical notation to abbreviate a functor to a single symbol F, and denote both maps in the ordered pair as F itself. To show that two functors are equal, it suffices to show that the maps bringing objects to objects and morphisms to morphisms coincide. In this manner we obtain the following proposition.

Proposition 27. *The following statements are in order.*

1. *If F is a covariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a contravariant functor from $\mathbb{A}^{op}$ to $\mathbb{B}$, and conversely.*

2. *If F is a covariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a contravariant functor from $\mathbb{A}$ to $\mathbb{B}^{op}$, and conversely.*

3. *If F is a covariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a covariant functor from $\mathbb{A}^{op}$ to $\mathbb{B}^{op}$, and conversely.*

4. *If F is a contravariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a covariant functor from $\mathbb{A}^{op}$ to $\mathbb{B}$, and conversely.*

5. *If F is a contravariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a covariant functor from $\mathbb{A}$ to $\mathbb{B}^{op}$, and conversely.*

6. *If F is a contravariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a contravariant functor from $\mathbb{A}^{op}$ to $\mathbb{B}^{op}$, and conversely.*

Proof. We prove the forward implication in 1. Since the opposite category has the same objects and morphisms, we have that F takes objects from $\mathbb{A}^{\mathrm{op}}$ to $\mathbb{B}$, and morphisms $\mathbb{A}^{\mathrm{op}}$ to $\mathbb{B}$.

We have that $f : X \to Y$ in $\mathbb{A}$, then $f : Y \to X$, so $\overline{F}(f) : F(Y) \to F(X)$ in $\mathbb{B}^{\mathrm{op}}$. Preservation of identity is by definition. For preservation of operation, we have that $F(\beta \circ_{\mathbb{A}^{\mathrm{op}}} \alpha) = F(\alpha \circ_{\mathbb{A}} \beta) = F(\alpha) \circ_{\mathbb{B}} \overline{F}(\beta)$.

2 is essentially the same as 1. Using $(\mathbb{A}^{\mathrm{op}})^{\mathrm{op}} = \mathbb{A}$, 3 is derivable from 1 and 2; 4 and 5 are respectively derivable from 1 and 2; and 6 is derivable from 4 and 5. □

There is a point to be made here. Recall that when define the category of sets, it is not sufficient to only consider functions as a subset of some product of two sets $A \times B$, but we also needed to encode the information of the domain and codomain in order to make a function belong to exactly one hom-set. We need to do this with regard to functors as well when defining the category of locally small categories which we will denote as **LSmall** (other authors may denote this as **Cat**). So when considering functors as a morphism of categories, it would not be precise to state "If F is a covariant functor from $\mathbb{A}$ to $\mathbb{B}$, then it is a covariant functor from $\mathbb{A}^{\mathrm{op}}$ to $\mathbb{B}^{\mathrm{op}}$, and conversely", and so on. However when discussing a functor in general, there is no need for this distinction.

2.4 Isomorphism and Equivalence of Categories

Given categories C and D, we shall say that they are "isomorphic" iff there exists a functors $F : C \to D$ and $G : D \to C$ such that $G \circ F = id_C$ and $F \circ G = id_D$. It is easily verified that this is true iff both the map from objects of C to objects of D and the map from morphisms of C to morphisms of D is bijective. It is easily confirmed that an isomorphism of categories is fully faithful. Isomorphisms of cateogies satisfy reflexivity, symmetry, and transitivity. We shal write $C \approx D$ iff C and D are isomorphic categories.

We observe the following for categories C and D.

- $C \times D \approx D \times C$
- $(C \times D) \times E \approx C \times (D \times E)$

We say that a functor is a "weak equivalence" iff it is fully faithful and essentially surjective.

Say that a category C is weakly equivalent to category D iff there exists a weak equivalence $F : C \to D$. It is then easily verified that weak equivalence satisfies reflexivity, symmtery and transitivity.

Say that a functor $F : C \to D$ is a "strong equivalence" iff there exists $G : D \to C$ such that $G \circ F \approx id_C$ in $Fct(C, C)$ and $F \circ G \approx id_D$ in $Fct(D, D)$.

Say that a category C is strongly equivalent to category D iff there exists a strong equivalence $F : C \to D$. It is then easily verified that strong equivalence satisfies reflexivity, symmtery and transitivity.

An isomorphism of categories is both a weak and strong equivalence.

2.5 Category of Locally Small Categories

Proposition 28. *The collection of all locally small categories forms a category, with covariant functors between categories as its morphisms.*

As agreed above, we shall denote the category of locally small categories as **LSmall**.

Any fully faithful functor in the category of locally small categories is monic. More generally, we have the following statement.

Proposition 29. *If $F : D \longrightarrow E$ is fully faithful and $G : C \longrightarrow D$ is a functor, then $F \circ G \approx F \circ G'$ implies that $G \approx G'$.*

Proof. Suppose α is a morphism in C. The commutativity of the diagram

$$\begin{array}{ccc} F(G(X)) & \xrightarrow{\sim} & F(G'(X)) \\ \downarrow{\scriptstyle F(G'(\alpha))} & & \downarrow{\scriptstyle F(G'(\alpha))} \\ F(G(Y)) & \xrightarrow{\sim} & F(G'(Y)) \end{array}$$

implies the commutativity of the diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{\sim} & G'(X) \\ \downarrow{\scriptstyle G'(\alpha)} & & \downarrow{\scriptstyle G'(\alpha)} \\ G(Y) & \xrightarrow{\sim} & G'(Y) \end{array}$$

because F is fully faithful. $\square$

When denoting the composition of two functors, we may write GF instead of $G \circ F$. Further, as shorthand, we may write FGX to denote $F(G(X)) = F \circ G(X)$ for object X, and $FG\alpha$ to denote $F(G(\alpha)) = F \circ G(\alpha)$ for morphism α.

Given morphism $F : C \to D$ in **LSmall**, denote F^{op} as the association from objects of C^{op} to D^{op} and morphisms from C^{op} to D^{op} by

$$X \mapsto FX$$

$$f \mapsto Ff$$

It is verified that $F^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$ is a morphism in **LSmall**. In the category **LSmall**, we observe that the dual of the covariant functor F is the covariant functor F^{op}. We may refer to F^{op} as the "cofunctor".

2.6 Category of Functors (Functor Categories)

Here we define what we mean by morphism or functors, or "natural transformations".

Definition 30. Given two categories C and D, the category of all covariant functors from C to D, denoted $\text{Fct}(C, D)$, $\text{Fun}(C, D)$, $[C, D]$, or C^D is the category whose objects are all functors from C to D. A morphism θ from functor F to functor G is a function from $Ob(C)$ to $Hom(D)$ such that:

1. $\theta(X) : F(X) \to G(X)$ for all objects $X \in Ob(C)$.
2. For all $\alpha : X \to Y$ in $Hom(C)$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\theta(X)"] \arrow[d, "F(\alpha)"] & G(X) \arrow[d, "G(\alpha)"] \\ F(Y) \arrow[r, "\theta(X)"'] & G(Y) \end{tikzcd}$$

   commutes.

It is then verified that defining $\eta \circ \theta(X) := \eta(X) \circ \theta(X)$ gives a law of composition of two morphisms of covariant functors such that $\mathrm{Fct}(C, D)$ is a category. Indeed, the identity morphism of functor F is precisely the map θ which takes each X to $id_{F(X)}$.

$\mathrm{Fct}(C, D)$ is called the category of functors from C to D. Category of this type are called "functor categories".

Given a morphism of functors θ, for shorthand, one may denote $\theta_X := \theta(X)$.

A morphism of functors is called a "natural transformation". Suppose F and G are functors. If we, in general, give some morphism (in particular isomorphism) $F(X) \to G(X)$ for some arbitrary object X, we shall say that this morphism is "natural in X" iff the diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "F(\alpha)"] & G(X) \arrow[d, "G(\alpha)"] \\ F(Y) \arrow[r] & G(Y) \end{tikzcd}$$

commutes for all X, that is, we are able to define a natural transformation with the given morphisms.

In particular, $Fct(C, \mathbf{Set})$ is called the category of "presheaves of C". This is not to be confused with the geometric definition of the category of presheaves.

To understand the category of functors and understand why we define morphisms between functors in such a way, we first consider a concrete example; two vector spaces. The first analogy that we may draw is that of a linear map; a linear map preserves the operation of elements; and in the same sense, when we look at applying a functor as an operation, we see the morphism of functors preserves this operation.

In a different specific analogy, suppose we have an isomorphism of vector spaces $F : V \to W$ and linear map α. Then the diagram

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[d, "\alpha"] & W \arrow[d, "F\alpha F^{-1}"] \\ V \arrow[r, "\sim"] & W \end{tikzcd}$$

commutes. Then the "corresponding" linear map of α in the space W is systematically given by $F\alpha F^{-1}$. So in the sense that W and V are in the sense, the same, α in V is the same as $F\alpha F^{-1}$ in W, and in fact the property that characterizes "sameness" is the commutativity of the above diagram. There may be various ways to define the isomorphism of the two spaces and therefore there may be multiple ways to give the corresponding linear map in W depending on how we view W. Looking at α as an operation in V, we have that $F\alpha F^{-1}$ as the corresponding operation in W. In a similar way, we expect a morphism of functors to obey commutativity for every morphism in the original categories.

Remark 31. When considering a morphism of functors, it is important to remember which category at hand is the one which we are treating. For example, just as if we are given two groups G and G', a morphism in **Set** may not necessarily be a morphism in **Grp** even though G and G' are objects of both categores. Similarly, even though a functor $F : C \to D$ may belong in $Fct(C, D)$ and $Fct(C^{\mathrm{op}}, D^{\mathrm{op}})$, we need to be careful which category we are dealing with when speaking of morphisms, and in particular, isomorphisms.

2.7 Evaluation Functor

Given categories C and D, we shall describe what is called an "evaluation functor". Given object $b \in Ob(B)$, write E_b for the functor

$$E_b : Fct(B, C) \to C$$

$$E_b : F \mapsto F(b)$$

$$E_b : \alpha \mapsto \alpha(b)$$

for functor $F : B \to C$, and morphism of functors $\alpha : F \to G$. We see that the identity and composition is respected.

Functors in $(Fct(B, C), C)$ of this kind are called "evaluation functors".

2.8 Composing Morphisms of Functors with Functors

Suppose we have morphism of functors $\theta : F \Longrightarrow G$ in $Fct(\mathbb{A}, C)$. Suppose $L : C \to D$; then we shall denote the morphism of functors $L\theta$ as the map which takes $A \in Ob(\mathbb{A})$ to the morphism $L(\theta_A)$ in D. Then since L is a functor, the commutativity of

$$\begin{tikzcd} F(A) \arrow[r, "\theta_A"] \arrow[d, "F(\alpha)"] & G(A) \arrow[d, "G(\alpha)"] \\ F(B) \arrow[r, "\theta_B"] & G(B) \end{tikzcd}$$

in C immediately results in the commutativity of

$$\begin{tikzcd} LF(A) \arrow[r, "L\theta_A"] \arrow[d, "LF(\alpha)"] & LG(A) \arrow[d, "LG(\alpha)"] \\ LF(B) \arrow[r, "L\theta_B"] & LG(B) \end{tikzcd}$$

in D, which shows that $L\theta : LF \Longrightarrow LG$ in $Fct(\mathbb{A}, D)$.

Also, suppose that we have $\theta : F \Longrightarrow G$ in $Fct(C, \mathbb{A})$. Suppose $R : D \to C$; then we shall denote the morphism of functors θR as the map which takes $X \in Ob(D)$ to the morphism $\theta_{R(X)}$ in $\mathbb{A}$. Then given $\alpha : X \to Y$ in C, we have the commutativity of

$$\begin{tikzcd} FR(X) \arrow[r, "\theta_{R(X)}"] \arrow[d, "FR(\alpha)"] & GR(X) \arrow[d, "GR(\alpha)"] \\ FR(Y) \arrow[r, "\theta_{R(Y)}"] & GR(Y) \end{tikzcd}$$

which shows that $\theta R : FR \Longrightarrow GR$.

Proposition 32. *If we have functors F and G in $Fct(C, D)$, such that $F \approx G$, then:*

- *For any L in $Fct(D, E)$, we have isomorphism $LF \approx LG$.*
- *For any R in $Fct(E, C)$, we have isomorphism $FR \approx GR$.abc (is this true?)*

2.9 Properties Regarding Isomorphism of Functors

Proposition 33. *If $F \approx F'$ in the category $Fct(C^{op}, D^{op})$, then $F \approx F'$ in the category $Fct(C, D)$.*

Proof. Sufficiently obvious. □

Proposition 34. $Fct(C, D)^{op} = Fct(C^{op}, D^{op})$.

Proof. abc □

Suppose we have categories I, J, and C, and functor $F : I \times J \to C$. Denote $F(\bullet, \bullet)$ as the object in $\mathrm{Fct}(I, \mathrm{Fct}(J, \mathcal{C}))$ that associates

$$i \mapsto F(\bullet, i)$$

$$\alpha \mapsto F(\bullet, \alpha)$$

and by the previous discussion on

Proposition 35. *A morphism of functors θ is an isomorphism iff its image consists only of isomorphisms. That is, $\theta(X)$ is an isomorphism for all X.*

Proof. Routine verification. □

Example 36. Given category C, and id_C in $Fct(C, C)$, we have that $End(id_C)$ is a commutative monoid.

Proof. We have that $End(id_C)$ is a monoid. To show commutativity, suppose $\theta, \eta \in End(id_C)$. Suppose $X \in Ob(C)$. Then since $\theta_X : X \to X$ is a morphism of C. Therefore the commuatative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "id(\theta_X)"] & X \arrow[d, "id(\theta_X)"] \\
X \arrow[r, "\eta_X"'] & X
\end{tikzcd}$$

which shows that $\theta_X \eta_X = \eta_X \theta_X$. □

2.10 Bifunctors

A functor F is said to be a "bifunctor" or "2-ary functor" when its source is a product category of two categories. Given covaraint bifunctor $F : C \times D \to \mathbb{A}$, and object $X \in Ob(C)$, define the functor $F(X, \bullet)$, also denoted $F(X, -)$ which takes C' to D, by associating $X' \in Ob(C')$ to the object $F(X, X')$ in D, and morphism $\alpha : X' \to Y'$ in C' to the morphism $F(id_X, \alpha) : F(X, X') \to$

$F(X, Y')$ in D. It is immediately verified that this makes $F(X, \bullet)$ a covariant functor.

Further, given covariant bifunctor $F : C \times D \to \mathbb{A}$, and object $Y \in Ob(D)$, define the functor

$$F(\bullet, Y) : D \longrightarrow \mathbb{A}$$

also denoted $F(-, Y)$, by associating $X \in Ob(C)$ to the object $F(X, Y)$ in D, and morphism $\beta : A \to B$ in C to the morphism $F(\beta, id_Y) : F(A, Y) \to F(B, Y)$ in D. It is immediately verified that this makes $F(\bullet, Y)$ a covariant functor.

Remark 37. We make some obvious remarks which, notwithstanding, are important to keep in mind. We are able to consider n-ary functors, an in general, a functor whose source is a product of any arbitrary collection of categories. We are also able to reason with induced functors $F(A_1, \cdots, A_{m-1}, \bullet, A_{m+1}, \cdots, A_n)$ and so on. When we reason in general about the functor $F(X, \bullet)$, we therefore reason about $F(\bullet, Y)$, and in general, $F(A_1, \cdots, A_{m-1}, \bullet, A_{m+1}, \cdots, A_n)$. Therefore giving the definition of $F(\bullet, X')$ was redundant.

Given covariant bifunctor $F : C \times D \to \mathbb{A}$, and morphism $\alpha : A \to B$ in D, we are able to define the morphism of functors

$$F(\bullet, \alpha) : F(\bullet, A) \longrightarrow F(\bullet, B)$$

which is the map which associates $X \in C$ to $F(X, \alpha) = F(id_X, \alpha)$. Then indeed, given any morphism $\eta : X \to Y$ in C, the diagram

$$\begin{tikzcd} F(\bullet, A)(X) \arrow[r, "F(X,\alpha)"] \arrow[d, "F(\eta,A)"] & F(\bullet, B)(X) \arrow[d, "F(\eta,B)"] \\ F(\bullet, A)(Y) \arrow[r, "F(Y,\alpha)"] & F(\bullet, B)(Y) \end{tikzcd}$$

obviously commutes.

Therefore it transpires that given covariant bifunctor $F : C \times D \to \mathbb{A}$, we are able to define the functor $\Xi^F : D \longrightarrow Fct(C, \mathbb{A})$ by the associations

$$\Xi^F : A \mapsto F(\bullet, A)$$

$$\Xi^F : \alpha \mapsto F(\bullet, \alpha)$$

for indeed, if $\alpha = id_A$, then $F(X, id_A)$ is identity of $F(X, A)$, and given $\beta \circ \alpha \in D$, we have that $F(X, \beta \circ \alpha) = F(X, \beta) \circ F(X, \alpha)$.

Now suppose we have morphism of functors $\theta : F \Longrightarrow G$. Denote $\theta(\bullet, A)$ as the map which associates

$$X \mapsto \theta(X, A)$$

which we see is a morphism from $F(\bullet, A) \to G(\bullet, A)$. To see that it is functorial in A, we observe the commutativity of

$$\begin{tikzcd} F(\bullet, A) \arrow[r, "\theta(\bullet,A)"] \arrow[d, "F(\bullet,\eta)"] & G(\bullet, A) \arrow[d, "G(\bullet,\eta)"] \\ F(\bullet, B) \arrow[r, "\theta(\bullet,B)"] & G(\bullet, B) \end{tikzcd}$$

in the category $Fct(C, \mathbb{A})$; by substituting X into the diagram, we see $G(\bullet, \eta) \circ \theta(\bullet, A)(X) = \theta(X, A) \circ G(X, \eta) = F(\bullet, \eta) \circ \theta(X, B)$.

So we are able to define an morphism of functors $\Xi^\theta : \Xi^F \to \Xi^G$ by mapping

$$\Xi^\theta : X \to \theta(\bullet, X)$$

and we see that the association

$$\Xi : F \mapsto \Xi^F$$

$$\Xi : \theta \mapsto \Xi^\theta$$

gives a functor $\Xi : Fct(C \times D, \mathbb{A}) \to Fct(D, Fct(C, \mathbb{A}))$. For indeed, we have

$$\Xi^{\gamma \circ \theta}(X)(A) = \gamma \circ \theta(\bullet, X)(A) = \gamma(A, X) \circ \theta(A, X) = \gamma(\bullet, X) \circ \theta(\bullet, X)(A)$$

$$\Xi^\gamma \circ \Xi^\theta(X)(A)$$

It is then a routine verification to show that Ξ is in fact an isomorphism of categories. Therefore we have

$$Fct(C \times D, \mathbb{A}) \approx Fct(D, Fct(C, \mathbb{A}))$$

Further, we also have the natural isomorphism of categories $s : D \times C \to C \times D$. One easily sees that we have an isomorphism of categories

$$Fct(C \times D, \mathbb{A}) \approx Fct(D \times C, \mathbb{A})$$

$$F \mapsto s \circ F$$

$$\theta \mapsto s \circ \theta(\bullet)$$

Therefore we also have a symmetric isomorphism

$$Fct(D \times C, \mathbb{A}) \to Fct(D, Fct(C, \mathbb{A}))$$

Definition 38. The Hom functor is a bifunctor of importance. Given locally small category C, denote $\mathrm{Hom}(\bullet, \bullet)$ as the Hom functor from $C^{\mathrm{op}} \times C$ to **Set**.

Given object (X, Y) in $C^{\mathrm{op}} \times C$, associate this to $\mathrm{Hom}(X, Y)$ which is a set. Then given morphism $(\alpha, \beta) : (X, Y) \to (X', Y')$ in $C^{\mathrm{op}} \times C$, we have that $(\alpha, \beta) : (X', Y) \to (X, Y')$ in $C \times C$; denote $\mathrm{Hom}(\alpha, \beta) := \beta \circ \bullet \circ \alpha$ as the function mapping $f \in \mathrm{Hom}(X, Y)$ to $\beta \circ f \circ \alpha \in \mathrm{Hom}(X', Y')$, which is the composition of the morphisms in C. With this we see that $\mathrm{Hom}(\bullet, \bullet)$ conserves identity and composition and is therefore a functor.

When we discuss multiple categories, it will be useful to write $\mathrm{Hom}_C(\bullet, \bullet)$ for $\mathrm{Hom}(\bullet, \bullet)$.

Denote $\mathrm{Hom}(X, \bullet)$ as the covariant functor induced by $\mathrm{Hom}(\bullet, \bullet)$ in its second argument, and $\mathrm{Hom}(\bullet, Y)$ as the covariant functor induced by $\mathrm{Hom}(\bullet, \bullet)$ in its first argument. Then we have that

$$\mathrm{Hom}(\bullet, Y)(\alpha) = \circ\alpha$$
$$\mathrm{Hom}(X, \bullet)(\beta) = \beta\circ$$

Proposition 39. *Given locally small category C, and $X \in Ob(C) = Ob(C^{op})$, the functors*

$$Hom_C(X, \bullet) : C \longrightarrow \mathbf{Set}$$

$$Hom_{C^{op}}(\bullet, X) : C \longrightarrow \mathbf{Set}$$

are the same.

Proof. Suppose $Y \in Ob(C^{\mathrm{op}})$. Then clearly the objects map to the same elements in **Set**. For $\alpha : Y \to Y'$ in C, we have that $\mathrm{Hom}_C(X, \alpha)$ maps $f \in \mathrm{Hom}_C(X, Y)$ to $\alpha \circ f \in \mathrm{Hom}_C(X, Y')$.

Then $\mathrm{Hom}_{C^{\mathrm{op}}}(\alpha, X)$ maps $f \in \mathrm{Hom}_{C^{\mathrm{op}}}(Y, X)$ to $f \circ_{\mathrm{op}} \alpha \in \mathrm{Hom}_{C^{\mathrm{op}}}(Y', X)$. This is exactly the same as saying that it maps $f \in \mathrm{Hom}_C(X, Y)$ to $\alpha \circ f \in \mathrm{Hom}_C(X, Y')$.

Therefore the functors are equal. $\square$

For covariant bifunctors $F : C \times D \to \mathbb{A}$ and $G : C \times D \to \mathbb{A}$, and transformation of functors θ from F to G, then we are able to uniquely associate the morphism of functors

$$\theta(\bullet, Y) : F(\bullet, Y) \longrightarrow G(\bullet, Y)$$

$$\theta(X, \bullet) : F(X, \bullet) \longrightarrow G(X, \bullet)$$

given any $Y \in Ob(D)$ and $X \in Ob(C)$. Obviously, we simply map $\theta(\bullet, Y)(X) = \theta(X, Y)$, and $\theta(X, \bullet)(Y) = \theta(X, Y)$.

Given categories C, D and $\mathbb{A}$. Given $Y \in Ob(D)$, denote Γ^Y as the functor which maps from $Fct(C \times D, \mathbb{A})$ to $Fct(C, \mathbb{A})$ which maps

$$F \mapsto F(\bullet, Y)$$

$$\theta \mapsto \theta(\bullet, Y)$$

for $F \in Ob(Fct(C \times D, \mathbb{A}))$ and $\theta \in Hom(Fct(C \times D, \mathbb{A}))$. For if θ is identity, then $\theta(\bullet, Y)(X) = \theta(X, Y)$ is identity of $F(X, Y) = F(\bullet, Y)(X)$ in $\mathbb{A}$. Further, we have

$$\begin{aligned} \theta \circ \eta(\bullet, Y)(X) &= \theta \circ \eta(X, Y) = \theta(X, Y) \circ \eta(X, Y) \\ &= \theta(\bullet, Y)(X) \circ \eta(\bullet, Y)(X) = \theta(\bullet, Y) \circ \eta(\bullet, Y)(X) \end{aligned}$$

which shows that Γ^Y is a functor.

Given $\alpha : Y \to Z$ in D, in the category $Fct(Fct(C \times D, \mathbb{A}), Fct(C, \mathbb{A}))$, define the map Γ^α as the map which associates

$$\Gamma^\alpha : F \mapsto F(\bullet, \alpha)$$

for any $F \in Fct(C \times D, \mathbb{A})$. Then we verify that Γ^α is a morphism of the functors Γ^Y to Γ^Z. For indeed, we see that the diagram

$$\begin{tikzcd} \Gamma^Y(F) \arrow[r, "\Gamma^\alpha(F)"] \arrow[d, "\Gamma^Y(\theta)"] & \Gamma^Z(F) \arrow[d, "\Gamma^Z(\theta)"] \\ \Gamma^Y(G) \arrow[r, "\Gamma^\alpha(G)"] & \Gamma^Z(G) \end{tikzcd}$$

is precisely the diagram

$$\begin{tikzcd}
F(\bullet,Y) \arrow[r, "F(\bullet,\alpha)"] \arrow[d, "\theta(\bullet,Y)"] & F(\bullet,Z) \arrow[d, "\theta(\bullet,Y)"] \\
G(\bullet,Y) \arrow[r, "G(\bullet,\alpha)"] & G(\bullet,Z)
\end{tikzcd}$$

which is a diagram of morphism of functors of $Fct(C, \mathbb{A})$. To verify that the diagram commutes therefore, we simply substitute aribtrary $X \in Ob(D)$. Then

$$\begin{tikzcd}
F(X,Y) \arrow[r, "F(X,\alpha)"] \arrow[d, "\theta(X,Y)"] & F(X,Z) \arrow[d, "\theta(X,Y)"] \\
G(X,Y) \arrow[r, "G(X,\alpha)"] & G(X,Z)
\end{tikzcd}$$

commutes, because θ is a morphism from F to G. So we conclude that Γ^{α} is a morphism.

Given categories C, D and $\mathbb{A}$, define the functor $\Gamma : Fct(D) \to Fct(Fct(C \times D, \mathbb{A}), Fct(D, \mathbb{A}))$ which maps

$$Y \mapsto \Gamma^{Y}$$

$$\alpha \mapsto \Gamma^{\alpha}$$

Then Γ indeed brings id_Y to the map $\Gamma^{id_Y} : F \mapsto F(\bullet, id_Y)$. This is the identity morphism of $F(\bullet, Y)$. Further, if $\beta \circ \alpha$ is a morphism in D, then we have that $\Gamma^{\beta\circ\alpha} : F \mapsto F(\bullet, \beta \circ \alpha) = F(\bullet, \beta) \circ F(\bullet, \alpha)$.

Proposition 40. *If the bifunctor $F : C \times D \longrightarrow \mathbb{A}$ is faithful, then $F(\bullet, X')$ and $F(X, \bullet)$ are faithful.*

Proof. Suppose that $X' \in Ob(D)$. We have that $F(\bullet, X') : Hom_C(X, Y) \longrightarrow Hom_{\mathbb{A}}(F(X, X'), F(X, Y'))$.

Suppose that $\alpha : X \to Y$ and $\beta : X \to Y$ are morphisms in C. Then $F(\bullet, X')(\alpha) = F(\alpha, id_{X'}) = F(\beta, id_{X'}) = F(\bullet, X')(\beta)$ implies $\alpha = \beta$. $\square$

Remark 41. That F is full does not necessarily imply that $F(\bullet, X')$ and $F(X, \bullet)$ are full.

Proposition 42. *For covariant bifunctors $F : C \times D \to \mathbb{A}$ and $G : C \times D \to \mathbb{A}$, if F and G are isomorphic in $Fct(C \times D, \mathbb{A})$, then we have induced functor isomorphisms*

$$F(\bullet, Y) \approx G(\bullet, Y)$$

$$F(X, \bullet) \approx G(X, \bullet)$$

for all $Y \in Ob(D)$ and $X \in Ob(C)$.

Proof. Sufficiently obvious.

□

2.11 Argument-wise Composition of Functors with Bifunctors

Given covariant bifunctor $F : C \times D \to \mathbb{A}$, and covariant functors $G : C' \to C$, $H : D' \to D$, define the bifunctor

$$F(G\bullet, H\bullet) : C' \times D' \longrightarrow \mathbb{A}$$

by mapping (X, Y) to $F(G(X'), H(G')) \in Ob(\mathbb{A})$, and mapping morphisms $(\alpha, \beta) : (X, Y) \to (X', Y')$ in $C' \times D'$ to $F(G(\alpha), H(\beta))$. Then it is immediately verified that $F(G\bullet, H\bullet)$ respects identity and composition and hence is a functor.

Further, when we have functor morphisms $(\theta, \eta) : (G_1, H_1) \longrightarrow (G_2, H_2)$ in the category $Fct(C', C) \times Fct(D', D)$, we are able to naturally define the morphism of functors

$$F(\theta, \eta) : F(G_1\bullet, H_1\bullet) \Longrightarrow F(G_2\bullet, H_2\bullet)$$

which maps (X, Y) to the morphism $F(\theta(X), \eta(Y))$. Checking commutativity of the desired square is easy.

In fact, we observe that we have a functor

$$Fct(C', C) \times Fct(D', D) \longrightarrow Fct(Fct(C' \times C, \mathbb{A}), Fct(D' \times D, \mathbb{A}))$$

$$(G, H) \mapsto F(G\bullet, H\bullet)$$

$$(\theta, \eta) \mapsto F(\theta, \eta)$$

In particular, if (θ, η) is an isomorphism, then we have that $F(\theta, \eta)$ is an isomorphism of functors.

Proposition 43. *For locally small category C, Given bifunctor $Hom_C : C^{op} \times C \to \mathbb{A}$ and functor $G : C' \to C$, consider bifunctor $F(\bullet, G\bullet)$. Then for $Y \in C'$, we have the equality*

$$\Gamma^Y(Hom_C(\bullet, G\bullet)) = \Gamma^{G(Y)}(Hom_C(\bullet, \bullet))$$

Proof. We easily see that these are both functors from C to $\mathbb{A}$. Then for $X \in C$, we have

$$\Gamma^Y(\mathrm{Hom}_C(\bullet, G\bullet))(X) = \mathrm{Hom}_C(\bullet, G\bullet)(X, Y) = \mathrm{Hom}(X, G(Y))$$

which coincides with $\Gamma^{G(Y)}(\mathrm{Hom}_C(\bullet, \bullet))(X)$. Further, for morphism α, we have

$$\Gamma^Y(\mathrm{Hom}_C(\bullet, G\bullet))(\alpha) = \mathrm{Hom}_C(\bullet, G\bullet)(\alpha, id_Y)$$

$$= \mathrm{Hom}(\alpha, G(id_Y)) = \mathrm{Hom}(\alpha, id_{G(Y)})$$

which coincides with $\Gamma^{G(Y)}(\mathrm{Hom}_C(\bullet, \bullet))(\alpha)$. □

Given categories C, C', D, D', $\mathbb{A}$, if $\sqsupset$, $\urcorner$ are objects of $Fct(C \times D, \mathbb{A})$, and $F : C' \to C$, $G : D' \to D$. Then given morphism $\theta : \sqsupset \to \urcorner$, we clearly see that $\theta(F\bullet, G\bullet)$ which maps $(X, Y) \in Ob(C' \times D')$ to $\theta(F(X), G(Y))$ is a morphism from $\sqsupset(F\bullet, G\bullet)$ to $\urcorner(F\bullet, G\bullet)$.

Proposition 44. *For categories C, C', D, D', $\mathbb{A}$, if $\sqsupset$, $\urcorner$ are objects of $Fct(C \times D, \mathbb{A})$, and $F : C' \to C$, $G : D' \to D$, and $\sqsupset \approx \urcorner$, then*

$$\sqsupset(F\bullet, G\bullet) \approx \urcorner(F\bullet, G\bullet)$$

in $Fct(C' \times D', \mathbb{A})$.

Proof. Obvious. □

Proposition 45. *Given functors $F : C' \to C$, $F' : C'' \to C'$, $G : D' \to D$, $G' : D'' \to D'$, and bifunctor $\sqsupset : C \times D \to \mathbb{A}$, denote $\kappa : C' \times D' \to \mathbb{A}$ as the functor $\sqsupset(F\bullet, G\bullet)$. Then*

$$\kappa(F'\bullet, G'\bullet) = \sqsupset(F' \circ F\bullet, G' \circ G\bullet)$$

Proof. Sufficiently obvious. □

2.12 Various Examples

Example 46. Given k-algebra A, and k-module M, consider functor $\mathrm{Hom}_A(A, \bullet) : \mathbf{Mod}^L(A) \to \mathbf{Set}$. Given module M, we are able to view $\mathrm{Hom}_A(A, M)$ as a k-module.

Given $\alpha : M \to N$, denote $\mathrm{Hom}_A(A, \alpha)$ as the k-module homomorphism which maps

$$\mathrm{Hom}_A(A, M) \longrightarrow \mathrm{Hom}_A(A, N)$$

$$f \mapsto \alpha \circ f$$

And clearly this map respects identity and composition.

Therefore let us denote $\mathrm{Hom}_A(A, \bullet) : \mathbf{Mod}(A) \to \mathbf{Mod}(k)$ as the functor which maps

$$M \mapsto \mathrm{Hom}_A(A, M)$$
$$\alpha \mapsto \mathrm{Hom}_A(A, \alpha)$$

Denote $for : \mathbf{Mod}^R(A) \to \mathbf{Mod}(k)$ as the forgetful functor, which takes A-modules to their corresponding k-modules. Then we have the isomorphism of functors $\mathrm{Hom}_A(A, \bullet) \approx for$ by the isomorphism

$$\mathrm{Hom}_A(A, M) \longrightarrow for(M)$$

$$f \mapsto f(1_A)$$

Proposition 47. *The functors $Hom_A(A, \bullet)$ and for are isomorphic in the category $\mathbf{Mod}^L(A)$.*

Proof. Denote $\theta(M)$ as the k-module isomorphism which takes

$$\mathrm{Hom}_A(A, M) \longrightarrow for(M)$$
$$f \mapsto f(1)$$

We have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(A, M) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}_A(A,\alpha)"] & for(M) \arrow[d, "for(\alpha)"] \\
\mathrm{Hom}_A(A, N) \arrow[r, "\sim"] & for(N)
\end{tikzcd}$$

which commutes. $\square$

3 Adjunctions and So On

3.1 Yoneda Lemma

In this section, we touch on one of the important results of category theory.

Proposition 48. *Given locally small category C, and morphism $f : X \to Y$ in C (that is, $f : Y \to X$ in C^{op}), we have the equality $Hom_C(f, \bullet) = Hom_{C^{op}}(\bullet, f)$.*

Proof. We have that $\mathrm{Hom}_C(f, \bullet)$ is a morphism of functors in $Fct(C, \mathbf{Set})$ and $\mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, f)$ is a morphism of functors in $Fct(C, \mathbf{Set})$.

So both functors are a map from $Ob(C)$ to $Hom(\mathbf{Set})$. Suppose $Z \in Ob(C)$. Then

$$\mathrm{Hom}_C(f, Z) : \mathrm{Hom}_C(X, Z) \longrightarrow \mathrm{Hom}_C(Y, Z)$$

$$\mathrm{Hom}_C(f, Z) : \alpha \mapsto \alpha \circ f$$

$$\mathrm{Hom}_{C^{\mathrm{op}}}(Z, f) : \mathrm{Hom}_{C^{\mathrm{op}}}(Z, X) \longrightarrow \mathrm{Hom}_{C^{\mathrm{op}}}(Z, Y)$$

$$\mathrm{Hom}_{C^{\mathrm{op}}}(f, Z) : \alpha \mapsto f \circ_{\mathrm{op}} \alpha$$

And since $\mathrm{Hom}_C(X, Z) = \mathrm{Hom}_{C^{\mathrm{op}}}(Z, X)$, and $\alpha \circ f = f \circ_{\mathrm{op}} \alpha$, we have that the functions coincide; that is, $\mathrm{Hom}_C(f, \bullet) = \mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, f)$. $\square$

For locally small category C, denote

$$C^\wedge := Fct(C^{\mathrm{op}}, \mathbf{Set})$$

$$C^\vee := Fct(C, \mathbf{Set})$$

Denote

$$h^C : C \to C^\wedge$$

as the functor which associates $X \in Ob(C)$ to $\mathrm{Hom}(\bullet, X)$, which is an object of $C^\wedge$, and $\alpha \in Hom_{C^{\mathrm{op}}}(X, Y)$ to $\mathrm{Hom}(\bullet, \alpha)$. Then h^C is called the "contravariant Yoneda embedding", or simply the "Yoneda embedding".

Futher, denote

$$h_C : C^{\mathrm{op}} \to C^{\vee}$$

as the covariant functor which associates $X \in Ob(C^{\mathrm{op}})$ to $\mathrm{Hom}(X, \bullet)$, which is an object of $C^{\vee}$, and $\alpha \in \mathrm{Hom}_C(X, Y)$ to $\mathrm{Hom}(\alpha, \bullet)$. We also see that h_C is a covariant functor from C to $Fct(C^{\mathrm{op}}, \mathbf{Set}^{\mathrm{op}})$. Then h_C is called the "covariant Yoneda embedding", or the "Yoneda coembedding".

We consider two covariant bifunctors from $C^{\mathrm{op}} \times C^{\wedge}$ to $\mathbf{Set}$. Recalling that $\mathrm{Hom}_{C^{\wedge}}(\bullet, \bullet)$ is a covariant bifunctor from $(C^{\wedge})^{\mathrm{op}} \times C^{\wedge}$ to $\mathbf{Set}$. Because $h^C : C \to C^{\wedge}$, we have that $h^C : C^{\mathrm{op}} \to (C^{\wedge})^{\mathrm{op}}$. Then $\mathrm{Hom}_{C^{\wedge}}(h^C \bullet, \bullet) : C^{\mathrm{op}} \times C^{\wedge} \to \mathbf{Set}$.

Denote $\bullet(\bullet)$ as the association which takes $(X, A) \in Ob(C^{\mathrm{op}} \times C^{\wedge})$ to the set $A(X)$. Then for morphism $(\alpha, \eta) : (Y, A) \to (X, B)$, since $\eta$is a morphism of functors, we have that the following diagram commutes:

$$\begin{tikzcd}
A(Y) \arrow[r, "\eta(Y)"] \arrow[d, "A(\alpha)"] & B(Y) \arrow[d, "B(\alpha)"] \\
A(X) \arrow[r, "\eta(X)"] & B(X)
\end{tikzcd}$$

and thus define $\eta(\alpha) := \eta(X) \circ A(\alpha) = B(\alpha) \circ \eta(Y)$, which is a map from $A(Y)$ to $B(X)$.

Lemma 49. *(Yoneda Lemma, Contravariant version) Given locally small category C, the two functors*

$$Hom_{C^{\wedge}}(h^C \bullet, \bullet)$$

$$\bullet(\bullet)$$

are isomorphic in the category $Fct(C^{op} \times C^{\wedge}, \mathbf{Set})$ by the following bijection:

$\varphi : Hom_{C^{\wedge}}(h^C X, A) \to A(X)$ by $\varphi : \theta \mapsto \theta(X)(id_X)$.

Proof. We first show that φ is bijective. We give inverse map $\psi : A(X) \to \mathrm{Hom}_{C^{\wedge}}(h^C X, A)$ as follows. Given $s \in A(X)$, define a function θ from $Ob(C)$ to $Hom(\mathbf{Set})$ by associating $Y \in Ob(C)$ to the map which associates $f \in \mathrm{Hom}_C(Y, X) = h^C(Y)$ to $A(f)(s) \in A(Y)$. Then suppose we have $\alpha : Y' \to Y$ in C^{op}. We have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(Y,X) \arrow[r, "\theta(Y)"] \arrow[d, "\mathrm{Hom}_C(\alpha,X)"] & A(Y) \arrow[d, "A(\alpha)"] \\
\mathrm{Hom}_C(Y',X) \arrow[r, "\theta(Y')"] & A(Y')
\end{tikzcd}$$

which is easily verified to be commutative, as we note that A is a contravariant functor from C. Since α was arbitrary, we have that θ is indeed a morphism from $\mathrm{Hom}_C(\bullet, X)$ to A and is therefore in $\mathrm{Hom}_{C^\wedge}(h^C X, A)$. Then $\varphi \circ \psi = id_{A(X)}$ because if $s \in A(X)$, and θ is as constructed, we get $\theta(X)(id_X) = A(id_X)(s) = id_{A(X)}(s) = s$.

Conversely, we show that $\psi \circ \varphi = id_{\mathrm{Hom}_{C^\wedge}(h^C X, A)}$. For suppose that $\theta : h^C(X) \Longrightarrow A$. This maps to $s := \theta(X)(id_X)$. We show that $\psi(s)$ coincides with θ. Suppose $Y \in Ob(C^{\mathrm{op}})$. Then $\varphi(s)(Y)$ takes $f \in \mathrm{Hom}_C(Y, X) = \mathrm{Hom}_{C^{\mathrm{op}}}(X, Y)$ to $A(f)(s) = A(f)(\theta(X)(id_X)) = A(f) \circ \theta(X)(id_X)$. This coincides with $\theta(Y)(id_X \circ f) = \theta(Y)(f)$ due to the diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(X,X) \arrow[r, "\theta(X)"] \arrow[d, "\mathrm{Hom}_C(f,X)"] & A(X) \arrow[d, "A(f)"] \\
\mathrm{Hom}_C(Y,X) \arrow[r, "\theta(Y')"] & A(Y)
\end{tikzcd}$$

in **Set** being commutative. So $\theta = \varphi(s)$, and bijectivity is shown.

We now show that given any morphism $(\alpha, \eta) : (Y, A) \longrightarrow (X, B)$ in $C^{\mathrm{op}} \times C^\wedge$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{C^\wedge}(h^C Y, A) \arrow[r, "\varphi(Y,A)"] \arrow[d, "\mathrm{Hom}_{C^\wedge}(h^C\alpha,\eta)"] & A(Y) \arrow[d, "\eta(\alpha)"] \\
\mathrm{Hom}_{C^\wedge}(h^C X, B) \arrow[r, "\varphi(X,B)"] & B(X)
\end{tikzcd}$$

in the category **Set**. For suppose $f : h^C(Y) \Longrightarrow A$ is a $C^\wedge$ morphism. Going the lower path, we obtain $\left(\eta \circ f \circ h^C(\alpha)\right)(X)(id_X) = \eta(X)(f(X)(\alpha))$. Going the upper path obtains $\eta(X)(A(\alpha)(f(Y)(id_Y))) = B(\alpha)\left(\eta(Y)\left(f(Y)(id_Y)\right)\right)$.

Since f is a morphism of functors, we have that

$$\begin{tikzcd}
\mathrm{Hom}_C(Y,Y) \arrow[r, "f(Y)"] \arrow[d, "\mathrm{Hom}_C(\alpha,X)"] & A(Y) \arrow[d, "A(\alpha)"] \\
\mathrm{Hom}_C(X,Y) \arrow[r, "f(X)"] & A(X)
\end{tikzcd}$$

commutes, so $A(\alpha) \circ f(Y) = f(X) \circ h^C(Y)(\alpha)$, so $\eta(X)(A(\alpha)(f(Y)(id_Y))) = \eta(X)(f(X)(\alpha))$, so the values of the maps coincide and the diagram commutes and φ is indeed an isomorphism of functors. □

Remark. It should be noted that the Yoneda lemma is not necessarily named so because Yoneda proved or discovered it, but simply because MacLane named it so.

We shall call the functor h^C the "Yoneda embedding", and it is also denoted by the letter "y". To distinguish it from h_C it would be practical to call it the "contravariant Yoneda embedding".

Corollary 50. *The contravariant Yoneda embedding is fully faithful.*

Proof. Suppose $X, Y \in Ob(C)$. Define $A := \mathrm{Hom}(\bullet, Y) \in Fct(C^{\mathrm{op}}, \mathrm{Set}) = C^\wedge$. Then by the Yoneda lemma, we have the bijection

$$\psi : A(X) \longrightarrow \mathrm{Hom}_{C^\wedge}(h^C X, A)$$

that is,

$$\psi : \mathrm{Hom}_C(X, Y) \longrightarrow \mathrm{Hom}_{C^\wedge}(\mathrm{Hom}(\bullet, X), \mathrm{Hom}(\bullet, Y))$$

Therefore it suffices to show that h^C coincides with ψ. We have that

$$h^C : f \mapsto \mathrm{Hom}_C(\bullet, f)$$

For $Z \in Ob(C^{\mathrm{op}})$, the morphism of functors $\mathrm{Hom}_C(\bullet, f)$ brings Z to $\mathrm{Hom}_C(Z, f) = f\circ$.

On the other hand, given $f \in \mathrm{Hom}_C(X, Y) = A(X)$, we associate $\theta : h^C X \Longrightarrow h^C Y = A$, a morphism of functors associating $Z \in Ob(C^{\mathrm{op}})$ to the map $\mathrm{Hom}_C(Z, X) \to \mathrm{Hom}_C(Z, Y) = A(Z)$ by taking $\alpha : Z \to X$ in C to $A(\alpha)(f) = \mathrm{Hom}_C(\alpha, Y)(f) = f \circ \alpha$. Therefore the two maps coincide, and hence h^C is bijective. □

Corollary 51. *The covariant Yoneda embedding is fully faithful.*

Proof. We want to show bijectivity of the map

$$h_C : \mathrm{Hom}_{C^{\mathrm{op}}}(X, Y) \longrightarrow \mathrm{Hom}_{C^\vee}(\mathrm{Hom}(X, \bullet), \mathrm{Hom}(Y, \bullet))$$

$$h_C : f \mapsto \mathrm{Hom}_C(f, \bullet)$$

Substitute C^{op} in to C in the contravariant Yoneda embedding. Then

$$h^{C^{\mathrm{op}}} : \mathrm{Hom}_{C^{\mathrm{op}}}(X, Y) \longrightarrow \mathrm{Hom}_{(C^{\mathrm{op}})^\wedge}(\mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, X), \mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, Y))$$

$$h^{C^{\mathrm{op}}} : f \mapsto \mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, f)$$

and noting that $(C^{\mathrm{op}})^\wedge = C^\vee$, and $\mathrm{Hom}_C(X, \bullet) = \mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, X)$, and $\mathrm{Hom}_C(f, \bullet) = \mathrm{Hom}_{C^{\mathrm{op}}}(\bullet, f)$, we obtain that $h_C = h^{C^{\mathrm{op}}}$, and therefore bijectivity is shown. $\square$

3.2 Representable functors

For a functor $F : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$, and $X \in Ob(C)$, we state that F is "represented by X" iff F is isomorphic to $h^C(X)$ in the category $C^\wedge$.

Similarly, for a functor $F : C \longrightarrow \mathbf{Set}$, and $X \in Ob(C)$, we state that $F$is "represented by X" iff F is isomorphic to $h_C(X)$ in the category $C^\vee$.

A functor that is represented by some object is called "representable".

Proposition 52. *For functor $F : C^{op} \longrightarrow \mathbf{Set}$, and two isomorphisms $F \approx h^C(X)$, and $F \approx h^C(Y)$, there exists one and only one morphism $\alpha : X \to Y$ such that the diagram*

$$\begin{tikzcd} F \arrow[r, "\sim"] \arrow[dr, "\sim"'] & h^C(X) \arrow[d, "h^C(\alpha)"] \\ & h^C(Y) \end{tikzcd}$$

commutes.

Proof. h^C is fully faithful, so we can associate precisely one morphism $\alpha : X \to Y$ to the morphism $h^C(\alpha) : h^C(X) \to h^C(Y)$. $\square$

3.3 Adjunctions via Hom-Set Equivalence

There are many known ways to equivalently define an adjunction of two functors. In this set of notes, we shall define adjunction via hom-set equivalence.

Definition 53. (Adjoints) We shall define adjoints in the category **LSmall**.

For two locally small categories C and D, and covariant functors $F : C \to D$ and $G : D \to C$, we state that "F is left adjoint to G", or "F is a left adjoint of G", or "G is right adjoint to F" iff the two functors

$$\mathrm{Hom}_D(F(\bullet), \bullet) : C^{\mathrm{op}} \times D \longrightarrow \mathbf{Set}$$

$$\mathrm{Hom}_C(\bullet, G(\bullet)) : C^{\mathrm{op}} \times D \longrightarrow \mathbf{Set}$$

are isomorphic in the category $Fct(C^{\mathrm{op}} \times D, \mathbf{Set})$. (Note that we regard F as a functor from C^{op} to D^{op}). That is, we have the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_D(FX, A) \arrow[r, "\rho_{X,A}", "\sim"'] \arrow[d, "\mathrm{Hom}_D(F(\alpha),\beta)"] & \mathrm{Hom}_C(X, GA) \arrow[d, "\mathrm{Hom}_C(G(\alpha),\beta)"] \\
\mathrm{Hom}_D(FY, B) \arrow[r, "\rho_{Y,B}", "\sim"'] & \mathrm{Hom}_C(Y, GB)
\end{tikzcd}$$

for all $\alpha : X \to Y$ in C^{op} and $\beta : A \to B$ in D.

We shall say that a pair of functors (F, G) is an "adjunction" iff F is left adjoint to G. It is also conventional to write $(F \dashv G)$ to state that (F, G) is an adjuntion. Given arbitrary functor F, we shall say that "F is left adjoint" iff there exists G such that $(F \dashv G)$. Given arbitrary functor G, we shall say that "G is right adjoint" iff $(F \dashv G)$. When a functor is either left or right adjoint, we simply say that it is "adjoint" (it is, of course better to specify whether it is left or right).

Given categories C, D we state that a pair of covariant functors (F, G) is an "adjunction between C and D" iff:

1. $F : C \to D$ and $G : D \to C$
2. F is left adjoint to G

The notion of the left adjoint is dual to the notion of the right adjoint in the category **LSmall** in the following manner. Suppose that $F : C \to D$ in

LSmall is left adjoint. Then $F^{\mathrm{op}} : C^{\mathrm{op}} \to D^{\mathrm{op}}$ in **LSmall** is right adjoint. For we have the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D^{\mathrm{op}}}(A, F^{\mathrm{op}}X) \arrow[r, "\rho_{X,A}", "\sim"'] \arrow[d, "\mathrm{Hom}_{D^{\mathrm{op}}}(\beta, F^{\mathrm{op}}(\alpha))"] & \mathrm{Hom}_{C^{\mathrm{op}}}(G^{\mathrm{op}}A, X) \arrow[d, "\mathrm{Hom}_{C^{\mathrm{op}}}(\beta, G^{\mathrm{op}}(\alpha))"] \\
\mathrm{Hom}_{D^{\mathrm{op}}}(B, F^{\mathrm{op}}Y) \arrow[r, "\rho_{Y,B}", "\sim"'] & \mathrm{Hom}_{C^{\mathrm{op}}}(G^{\mathrm{op}}B, Y)
\end{tikzcd}$$

for all $\beta : A \to B$ in $(D^{\mathrm{op}})^{\mathrm{op}}$ and $\alpha : X \to Y$ in C^{op}, which means that $(G^{\mathrm{op}} \dashv F^{\mathrm{op}})$ and hence G^{op} is a left adjoint.

So: $(F \dashv G)$ implies $(G^{\mathrm{op}} \dashv F^{\mathrm{op}})$.

For this reason, we might formally call a right adjoint simply an "adjoint", and a left adjoint a "coadjoint" (why it is the case that we do not call the left adjoint the adjoint instead will be clear when discussing the limit and colimit and their preservation under adjoints). When we say that a functor is adjoint, we shall mean that it is right adjoint.

Proposition 54. *If F is left adjoint to G, then for all $Y \in Ob(D)$, $G(Y)$ is a representative of the functor $Hom_D(F(\bullet), Y)$.*

Further, if F is left adjoint to G, then for all $X \in Ob(C)$, $F(X)$ is a representative of the functor $Hom_D(X, G(\bullet))$.

Proof. This immediately follows from the definition that $\mathrm{Hom}_D(F(\bullet), \bullet)$ and $\mathrm{Hom}_C(\bullet, G(\bullet))$ are isomorphic. □

Proposition 55. *Adjoint functors are unique up to isomorphism.*

Proof. Suppose $G$and G' are both right adjoints of F. Suppose $\alpha : Y \to Y'$ in D.

Denote θ as the isomorphism $\mathrm{Hom}_C(\bullet, G(\bullet)) \approx \mathrm{Hom}_C(\bullet, G'(\bullet))$. Then $\theta(\bullet, Y)$ is the natural isomorphism between

$$h^C(G(Y)) = \mathrm{Hom}_C(\bullet, G(Y)) \approx \mathrm{Hom}_C(\bullet, G'(Y)) = h^C(G'(Y))$$

in the category $Fct(C^{\mathrm{op}}, \mathbf{Set})$. We would like to show that this isomorphism is natural in Y.

Recall that for any three categories C, D, and $\mathbb{A}$, we have defined a functor $\Gamma : Fct(D) \to Fct(Fct(C \times D, \mathbb{A}), Fct(D, \mathbb{A}))$. Given $\alpha : Y \to Z$ in D, in the category $Fct(Fct(C \times D, \mathbb{A}), Fct(C, \mathbb{A}))$, we have the morphism of functors from Γ^Y to Γ^Z, which maps functors F by

$$\Gamma^\alpha : \mathcal{F} \mapsto \mathcal{F}(\bullet, \alpha)$$

in which case, apply Γ^α to $\mathrm{Hom}_C(\bullet, G(\bullet))$ and $\mathrm{Hom}_C(\bullet, G'(\bullet))$.

We have the commutative diagram

$$\begin{tikzcd}
\Gamma^Y(\mathcal{F}) \arrow[r, "\Gamma^\alpha(\mathcal{F})"] \arrow[d, "\Gamma^Y(\theta)"] & \Gamma^Z(\mathcal{F}) \arrow[d, "\Gamma^Z(\theta)"] \\
\Gamma^Y(\mathcal{G}) \arrow[r, "\Gamma^\alpha(\mathcal{G})"] & \Gamma^Z(\mathcal{G})
\end{tikzcd}$$

Recalling that we have

$$\Gamma^Y(\mathrm{Hom}_C(\bullet, G\bullet)) = \Gamma^{G(Y)}(\mathrm{Hom}_C(\bullet, \bullet))$$

So therefore we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(\bullet, G(Y)) \arrow[r, "\mathrm{Hom}_C(\bullet, G(\alpha))"] \arrow[d, "\theta(\bullet, Y)"] & \mathrm{Hom}_C(\bullet, G(Z)) \arrow[d, "\theta(\bullet, Y')"] \\
\mathrm{Hom}_C(\bullet, G'(Y)) \arrow[r, "\mathrm{Hom}_C(\bullet, G(\alpha))"] & \mathrm{Hom}_C(\bullet, G'(Z))
\end{tikzcd}$$

Since h^C is fully faithful, we have the commutativity of

$$\begin{tikzcd}
G(Y) \arrow[r, "G(\alpha)"] \arrow[d, "\sim"] & G(Z) \arrow[d, "\sim"] \\
G'(Y) \arrow[r, "G'(\alpha)"] & G'(Z)
\end{tikzcd}$$

Since α was aribtrary, this shows $G \approx G'$.

Now obversely, if we suppose $\mathrm{Hom}_D(F(\bullet), \bullet) \approx \mathrm{Hom}_D(F'(\bullet), \bullet)$, we simply observe that

$$\mathrm{Hom}_{D^{\mathrm{op}}}(\bullet, \bullet) : D \times D^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

$$F : C^{\mathrm{op}} \longrightarrow D^{\mathrm{op}}$$

and therefore

$$\mathrm{Hom}_{D^{\mathrm{op}}}(\bullet, F(\bullet)) : (D^{\mathrm{op}})^{\mathrm{op}} \times C^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

Observing the equality of the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_D(F(X),Y) \arrow[r, "\theta(X,Y)"] \arrow[d, "\mathrm{Hom}_D(F(\alpha),\beta)"] & \mathrm{Hom}_D(F'(X),Y) \arrow[d, "\mathrm{Hom}_D(F'(\alpha),\beta)"] \\
\mathrm{Hom}_D(F(X'),Y') \arrow[r, "\theta(X',Y')"] & \mathrm{Hom}_D(F'(X'),Y')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{D^{\mathrm{op}}}(Y,F(X)) \arrow[r, "\theta(X,Y)"] \arrow[d, "\mathrm{Hom}_{D^{\mathrm{op}}}(\beta,F(\alpha))"] & \mathrm{Hom}_{D^{\mathrm{op}}}(Y,F'(X)) \arrow[d, "\mathrm{Hom}_{D^{\mathrm{op}}}(\beta,F'(\alpha))"] \\
\mathrm{Hom}_{D^{\mathrm{op}}}(Y',F(X')) \arrow[r, "\theta(X',Y')"'] & \mathrm{Hom}_{D^{\mathrm{op}}}(Y',F'(X'))
\end{tikzcd}$$

we immediately obtain that $F \approx F'$ in the category $Fct(C^{\mathrm{op}}, D^{\mathrm{op}})$ and hence $F \approx F'$ in the category $Fct(C, D)$. □

We also observe that due to properties regarding composition of bifunctors with functors arguementwise, if F is left adjoint to G, then if G is isomorphic to G', then F is also left adjoint to G'.

Proposition 56. *An isomorphism G of locally small categories is automatically an adjoint (and coadjoint) functor.*

Proof. Put F as the inverse functor of G. When we have isomorphism

$$\theta : \mathrm{Hom}_D(F(\bullet), F(\bullet)) \simeq \mathrm{Hom}_C(\bullet, G \circ F(\bullet)) = \mathrm{Hom}_C(\bullet, \bullet)$$

by associating $f : F(X) \to F(Y)$ to the functor $G(f)$.

By composing F^{-1} in the second argument, we get

$$\mathrm{Hom}_D(F^{-1}(\bullet), \bullet) \approx \mathrm{Hom}_C(\bullet, G(\bullet))$$

The coadjoint case is similarly proven. □

Definition 57. $F : C \to D$ and $G : D \to C$

3.4 Adjunct Morphisms

Suppose we have the adjoint situation $\rho : \mathrm{Hom}_D(F(\bullet), \bullet) \approx \mathrm{Hom}_C(\bullet, G(\bullet))$. Given objects $X \in Ob(C)$ and $A \in Ob(D)$, we have isomorphism

$$\rho : \mathrm{Hom}_D(F(X), A) \approx \mathrm{Hom}_C(X, G(A))$$

Given morphisms $f : F(X) \to A$ in D and $g : X \to G(A)$ in C, we shall say that f and g are "adjunct morphisms" iff $\rho_{X,A}(f) = g$. In this case, we say that "f is the left adjunct of g" and "g is the right adjunct of f" As a matter of notation, one may denote $f^{\#} := g$, and $g^{\flat} := f$. One may also denote $\widetilde{f} := g$.

Remark. Functors are adjoint; morphisms are adjunct.

3.5 Adjunction of Functors via Unit-Counit Adjunction

We shall first say what we mean by units and counits.

Proposition 58. *For functors $F : C \to D$ and $G : D \to C$ on locally small categories, if F is the left adjoint of G, then we have the isomorphism of functors*

$$\theta : Hom_D(F(\bullet), F(\bullet)) \simeq Hom_C(\bullet, G \circ F(\bullet))$$
$$\eta : Hom_D(F \circ G(\bullet), \bullet) \simeq Hom_C(G(\bullet), G(\bullet))$$

in the categories $Fct(D \times D, \mathbf{Set})$ and $Fct(C \times C, \mathbf{Set})$.

In particular, if $\rho : Hom_D(F(\bullet), \bullet) \approx Hom_C(\bullet, G(\bullet))$, then $\theta = \rho(\bullet, F\bullet)$ and $\eta = \rho(G\bullet, \bullet)$.

Proof. This is immediately derivable from previous remarks on composing bifunctors with functors in each argument. □

Definition 59. Given left adjoint $F : C \to D$ of functor $G : D \to C$, denote the isomorphism

$$\theta_X : \mathrm{Hom}_D(F(X), F(X)) \to \mathrm{Hom}_C(X, G \circ F(X))$$

we shall define a funtion which maps $Ob(C)$ to $\mathrm{Hom}(C)$ as follows.

Given $X \in Ob(C)$, have isomorphism in C

$$\theta_X : \mathrm{Hom}_D(F(X), F(X)) \to \mathrm{Hom}_C(X, G \circ F(X))$$

Then take X to id_X, and take id_X to $\theta_X(id_{F(X)})$. Map X to this element. Denote this map as γ. This is known as the "adjunction map" or (more commonly) the "unit" of the adjunction (F, G).

Then we show that this gives a morphism of functors from id_C to $G \circ F$ in the category $Fct(C, C)$. For suppose we have morphism $\alpha : X \to Y$ in C. Then consider the diagram

$$\begin{tikzcd} X \arrow[r, "\theta_X(id_{F(X)})"] \arrow[d, "\alpha"] & G \circ F(X) \arrow[d, "G\circ F(\alpha)"] \\ Y \arrow[r, "\theta_Y(id_{F(Y)})"'] & G \circ F(Y) \end{tikzcd}$$

To show that this commutes simply observe the commutativity of the diagram in **Set**

$$\begin{tikzcd} \mathrm{Hom}_D(F(X), F(X)) \arrow[r, "\theta_X"] \arrow[d, "\mathrm{Hom}_D(F(id_X), F(\alpha))"] & \mathrm{Hom}_C(X, G \circ F(X)) \arrow[d, "\mathrm{Hom}_C(id_X, G\circ F(\alpha))"] \\ \mathrm{Hom}_D(F(X), F(Y)) \arrow[r, "\theta_{XY}"] & \mathrm{Hom}_C(X, G \circ F(Y)) \end{tikzcd}$$

Evaluating $id_{F(X)}$, which is both a functor from D to D and D^{op} to D^{op}, we obtain the commutativity of

$$\begin{tikzcd} X \arrow[r, "\theta_X(id_{F(X)})"] \arrow[dr, "\theta_{XY}(F(\alpha))"'] & G \circ F(X) \arrow[d, "G\circ F(\alpha)"] \\ & G \circ F(Y) \end{tikzcd}$$

Similarly, since $\alpha : Y \to X$ in the category C^{op}, observe that we have commutativity of

$$\begin{tikzcd} \mathrm{Hom}_D(F(Y), F(Y)) \arrow[r, "\theta_Y"] \arrow[d, "\mathrm{Hom}_D(F(\alpha), F(id_Y))"] & \mathrm{Hom}_C(Y, G \circ F(Y)) \arrow[d, "\mathrm{Hom}_C(\alpha, G\circ F(id_Y))"] \\ \mathrm{Hom}_D(F(X), F(Y)) \arrow[r, "\theta_{XY}"] & \mathrm{Hom}_C(X, G \circ F(Y)) \end{tikzcd}$$

evaluating $id_{F(X)}$ we obtain the commutativity of

$$\begin{tikzcd} X \arrow[d, "\alpha"] \arrow[dr, "\theta_{XY}(F(\alpha))"] & \\ Y \arrow[r, "\theta_Y(id_{F(Y)})"'] & G \circ F(Y) \end{tikzcd}$$

which proves the commutativity of the desired square. Therefore γ is a morphism from id_C to $G \circ F$.

Remark. We can shorten the argument by arguing (in an informal way) as follows. We observe the commutativity of

$$\begin{tikzcd} F(X) \arrow[r, "id_{F(X)}"] \arrow[d, "F(\alpha)"] & F(X) \arrow[d, "F(\alpha)"] \\ F(Y) \arrow[r, "id_{F(Y)}"'] & F(Y) \end{tikzcd}$$

and hence applying θ, which is an isomorphism of functors, we get the desired commutative square.

Similarly, we are able to define a morphism of functors ε from $F \circ G$ to id_D by mapping

$$\varepsilon : A \mapsto \eta_A^{-1}(id_{G(A)})$$

for $A \in Ob(D)$.

We shall call ε the "coadjunction map", or (more commonly), the "counit" of the adjunction (F, G). It is convention to write $(F \dashv_\varepsilon^\gamma G)$ to say that γ is the unit and ε is the counit. When we write $F \dashv_\varepsilon^\gamma G : D \to C$, this gives more information: it means that $F : C \to D$ and $G : D \to C$ as well.

Suppose $(F \dashv_\varepsilon^\gamma G)$, and suppose $\rho : \mathrm{Hom}_D(F(\bullet), \bullet) \approx \mathrm{Hom}_C(\bullet, G(\bullet))$ is the explicit isomorphism. We note the following properties:

- $\gamma_X = \theta_X(id_{F(X)}) = \rho_{X,F(X)}(id_{F(X)})$ for $X \in Ob(C)$
- $\varepsilon_A = \eta_A^{-1}(id_{G(A)}) = \rho_{G(A),A}^{-1}(id_{G(A)})$ for $X \in Ob(C)$

Proposition 60. *We have formulas for expressing the adjunct of a morphism as functions of the adjoint functors and the unit/counit.*

Suppose $(F \dashv_\varepsilon^\gamma G)$. Then:

1. *If $f : F(X) \to A$ in C and $\widetilde{f} : X \to G(A)$ are adjunct morphisms, then $\widetilde{f} = G(f) \circ \gamma_X$*
2. *If $f : F(X) \to A$ in C and $\widetilde{f} : X \to G(A)$ are adjunct morphisms, then $f = \varepsilon_A \circ F(\widetilde{f})$*

Proof. Given $f : F(X) \to A$, by definition of ρ, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_D(F(X), F(X)) & \xrightarrow{\rho_{X,FX}} & \mathrm{Hom}_C(X, GF(X)) \\ \downarrow{\scriptstyle \mathrm{Hom}_D(F(id),f)} & & \downarrow{\scriptstyle \mathrm{Hom}_C(id,G(f))} \\ \mathrm{Hom}_D(F(X), A) & \xrightarrow{\rho_{X,A}} & \mathrm{Hom}_C(X, G(A)) \end{array}$$

commutes, and hence by evaluating $id_{F(X)}$, obtain 1.

Note that we have $\widetilde{f} : G(A) \to X$ in the opposite category C^{op}. Then we have the commutativity of

$$\begin{array}{ccc} \mathrm{Hom}_D(FG(A), A) & \xrightarrow{\rho_{GA,A}} & \mathrm{Hom}_C(G(A), G(A)) \\ \downarrow{\scriptstyle \mathrm{Hom}_D(F(\widetilde{f}),id)} & & \downarrow{\scriptstyle \mathrm{Hom}_C(\widetilde{f},G(id))} \\ \mathrm{Hom}_D(F(X), A) & \xrightarrow{\rho_{X,A}} & \mathrm{Hom}_C(X, G(A)) \end{array}$$

which, by evaluating $id_{G(A)}$, obtain 2. □

Proposition 61. *(Unit-Counit Adjunction) Suppose $(F \dashv_{\varepsilon}^{\gamma} G)$. Denote 1_F as the identity morphism of F and 1_G as the identity morphism on G. Denote $\varepsilon F \circ F\gamma$ as the map from Ob(C) to Hom(D) which takes X to $\varepsilon(F(X)) \circ F(\gamma(X))$. Denote $G\varepsilon \circ \gamma G$ as the map from Ob(D) to Hom(C) which takes A to $G(\varepsilon(A)) \circ \gamma(G(A))$.*

The following equalities are known as the "triangle identities" or "zig zag equations" or "unit counit adjunction"

1. $1_F = \varepsilon F \circ F\gamma$
2. $1_G = G\varepsilon \circ \gamma G$

Proof. We have that

$$id_{FX} = \rho^{-1}_{X,FX}\rho_{X,FX}(id_{FX}) = \rho^{-1}_{X,FX}(\gamma_X)$$

having that $\gamma_X : X \to GFX$, applying #2 of the previous proposition by putting $A = FX$, viewing γ_X as the right adjunct of $id_{F(X)}$, we get

$$id_{F(X)} = \varepsilon_{F(X)} \circ F(\gamma_X)$$

which proves 1. Again, from

$$id_{GA} = \rho^{-1}_{GA,A}\rho_{GA,A}(id_{GA}) = \rho^{-1}_{GA,A}(\varepsilon_A)$$

and having that $\varepsilon_A : FGA \to A$, applying #1 of the previous proposition, we get

$$id_{GA} = G(\varepsilon_A) \circ \gamma_{GA}$$

which proves 2. □

3.6 Examples of Adjoint Functors

Example 62. Consider the category of sets. Given set X, the functor $\bullet \times X$ is the left adjoint of the functor $\mathrm{Hom}(\bullet, X)$.

Proof. We would like to prove

$$\mathrm{Hom}(\bullet \times X, \bullet) \approx \mathrm{Hom}(\bullet, \mathrm{Hom}(\bullet, X))$$

in $Fct(\mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}, \mathbf{Set})$.

Given sets I, Y, we have isomorphism

$$\varphi : \mathrm{Hom}(I \times X, Y) \to \mathrm{Hom}(I, \mathrm{Hom}(X, Y))$$

which maps $f : I \times X \to Y$ to the map which takes $i \in I$ to the map $f(i, \bullet)$, which takes $x \in X$ to $f(i, x)$. We might denote this as $f(1, 2)$, where the numbers indicate the order of substitution.

Suppose $\alpha : I \to I'$ in $\mathbf{Set}^{\mathrm{op}}$, and $\beta : Y \to Y'$ in $\mathbf{Set}$.

$$\begin{tikzcd}
\mathrm{Hom}(I \times X, Y) \arrow[r, "\theta(I,Y)"] \arrow[d, "\mathrm{Hom}(\alpha \times X, \beta)"] & \mathrm{Hom}(I, \mathrm{Hom}(X, Y)) \arrow[d, "\mathrm{Hom}(\alpha, \mathrm{Hom}(X, \beta))"] \\
\mathrm{Hom}(I' \times X, Y') \arrow[r, "\theta(I',Y')"] & \mathrm{Hom}(I', \mathrm{Hom}(X, Y'))
\end{tikzcd}$$

Then we have $\mathrm{Hom}(\alpha \times X, \beta) = \beta \circ \bullet \circ \alpha \times id_X$, and $\mathrm{Hom}(\alpha, \mathrm{Hom}(\beta, X)) = \mathrm{Hom}(\beta, X) \circ \bullet \circ \alpha$. Suppose $f : I \times X \to Y$.

Go the lower path. We have that this brings $(i', x) \in I' \times X$ to

$$(i', x) \mapsto (\alpha(i'), b) \mapsto f(\alpha(i'), x) \mapsto \beta(f(\alpha(i'), x))$$

and then to the map which takes $i \in I'$ to the map which takes $x \in X$ to $\beta(f(\alpha(i'), x))$.

Go the upper path. We get $f(1,2)$. Apply the second morphism. Suppose $i' \in I'$. This is brought to $\alpha(i')$. Suppose $x \in X$. Then

$$\mathrm{Hom}(X,\beta)(f(\alpha(i'),\bullet)) = \beta \circ f(\alpha(i'),\bullet)$$

takes x to $\beta(f(\alpha(i'),x))$.

Therefore the diagram commutes and hence φ is an isomorphism which respect functoriality. Therefore this proves adjointness.

Given k-algebra A, and k-module L, consider functor $\mathrm{Hom}\,_k(L,\bullet) : \mathbf{Mod}^R(A) \to \mathbf{Mod}^R(A)$ which maps

$$M \mapsto \mathrm{Hom}\,_k(L,M)$$

$$\alpha \mapsto \mathrm{Hom}\,_k(L,\alpha)$$

for object M and module homomorphism $\alpha : M \to N$, where $\mathrm{Hom}\,_k(L,\alpha)$ denotes the application $\alpha\circ$ to all elements in $\mathrm{Hom}_k(L,M)$. $\square$

Consider the functor $L \otimes_k \bullet : \mathbf{Mod}^L(A) \to \mathbf{Mod}^L(A)$ which maps

$$M \mapsto L \otimes_k M$$

$$\alpha \mapsto L \otimes_k \alpha$$

for object M and A-module homomorphism $\alpha : M \to N$, where $L \otimes_k \alpha$ denotes the application

$$l \otimes_k m \mapsto l \otimes_k \alpha(m)$$

for all elements in $L \otimes_k M$, which is easily confirmed to be well defined, and we see that $L \otimes_k \bullet$ does indeed respect identity and composition.

Example 63. The functor $L \otimes_k \bullet$ is left adjoint of $\mathrm{Hom}\,_k(L,\bullet)$.

Proof. We desire to show the isomorphism

$$\mathrm{Hom}_A(L \otimes_k \bullet, \bullet) \approx \mathrm{Hom}_A(\bullet, \mathrm{Hom}_k(L,\bullet))$$

in $Fct(\mathbf{Mod}^L(A)^{\mathrm{op}} \times \mathbf{Mod}^L(A), \mathbf{Set})$. Given right A-modules N and M, recall that we have the canonical k-module homomorphism

$$\varphi : \mathrm{Hom}_A(L \otimes_k N, M) \to \mathrm{Hom}_A(N, \mathrm{Hom}_k(L,M))$$

whereby $\varphi(f)(n) = f(n, \bullet)$, which we may denote as $f(1,2)$. Suppose we have module isomorphisms $\alpha : N' \to N$ and $\beta : M \to M'$ in $\mathbf{Mod}^R(A)$. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(L \otimes_k N, M) \arrow[r, "\theta(N,M)"] \arrow[d, "\mathrm{Hom}_A(L\otimes_k \alpha,\beta)"] & \mathrm{Hom}_A(N, \mathrm{Hom}_k(L, M)) \arrow[d, "\mathrm{Hom}_A(\alpha,\mathrm{Hom}_k(L,\beta))"] \\
\mathrm{Hom}_A(L \otimes_k N', M') \arrow[r, "\theta(N',M')"] & \mathrm{Hom}_A(N', \mathrm{Hom}_k(L, M'))
\end{tikzcd}$$

Suppose $f \in \mathrm{Hom}_A(N \otimes_k L, M)$. Go the lower path. We first get a map that brings

$$\sum t(l \otimes n) \mapsto \sum t(l \otimes \alpha(n)) \mapsto \sum t \cdot f(l \otimes \alpha(n)) \mapsto \beta \left(\sum t \cdot f(l \otimes \alpha(n)) \right)$$

and then a map that brings $n' \in N'$ the map that takes $l \in L$ to the object $\beta\left(f(l \otimes \alpha(n'))\right)$.

Go the upper path. We first get $f(1,2)$. Apply the second morphism. Suppose $n' \in N'$. Then this maps to $\beta \circ (f(\bullet, \alpha(n'))$. This map takes $l \in L$ to $\beta(f(l, \alpha(n')))$. So the diagram commutes, and isomorphism is proven. $\square$

Denote $\bullet \otimes_k A : \mathbf{Mod}^R(A) \to \mathbf{Mod}(k)$ as the functor which takes

$$N \mapsto A \otimes_k N$$

$$\alpha \mapsto A \otimes_k \alpha$$

where $\alpha \otimes_k A$ denotes the application

$$n \otimes_k a \mapsto \alpha(n) \otimes_k a$$

Example 64. $\bullet \otimes_k A$ is the left adjoint of the functors $\mathrm{Hom}_A(A, \bullet)$ and for.

Proof. We desire to demonstrate

$$\mathrm{Hom}_A\left(\bullet \otimes_k A, \bullet\right) \approx \mathrm{Hom}_k(\bullet, \mathrm{Hom}_A(A, \bullet))$$

Recall that we have k-module isomorphism

$$\mathrm{Hom}_A\left(L \otimes_k A, M\right) \longrightarrow \mathrm{Hom}_k(L, \mathrm{Hom}_A(A, M))$$

$$f \mapsto f(1,2)$$

We therefore have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(L \otimes_k A, M) \arrow[r, "\theta(N,M)"] \arrow[d, "\mathrm{Hom}_A(\alpha \otimes_k A, \beta)"] & \mathrm{Hom}_k(L, \mathrm{Hom}_A(A, M)) \arrow[d, "\mathrm{Hom}_k(\alpha, \mathrm{Hom}_A(A,\beta))"] \\
\mathrm{Hom}_A(L \otimes_k A, M') \arrow[r, "\theta(N',M')"] & \mathrm{Hom}_k(L', \mathrm{Hom}_A(A, M'))
\end{tikzcd}$$

which is easily shown to be commutative with a proof similar to those as the above two examples. $\square$

4 Examples of Universal Objects

Various notions across categories can be unified by defining their properties in terms of universality in some category. In particular, when defining particular objects in a category, it is often the case that instead of focusing on the object itself, we instead focus on how the object needs to behave in relation to other objects. Such objects need to be indistingushable under isomorphism, even if there is a "most obvious way" to define such objects.

We recall that we are able to define the product of sets, the product of groups, rings and other algebraic objects, the product of topological spaces, and so on. To characterize what we mean by a "product" we introduce the concept of products in the most general sense. By doing so, we will also see that we can define notion of the empty product, that is, the product of zero objects in an arbitrary category.

4.1 Products and coproducts of objects in a category.

Example 65. Given category C, and a set of objects $S := \{X_i\}_{i\in I} \subset Ob(C)$, indexed by an arbitrary I (which is not necessarily small), consider the category $\mathbf{Prod}_C(S)$, which we define as follows.

Take the objects of $\mathbf{Prod}(S)$ to be the collection of all pairs $(Y, \{f_i\}_{i\in I})$, such that $Y \in Ob(C)$ and $f_j : Y \to X_j$. We shall call Y the "object part" of $(Y, \{f_i\}_{i\in I})$, and $\{f_i\}_i$ the "morphism part" of $(Y, \{f_i\}_{i\in I})$. Given two objects $(Y, \{f_i\}_{i\in I})$ and $(Y', \{g_i\}_{i\in I})$, say that $\alpha : Y \to Y'$ in C is a morphism from $(Y, \{f_i\}_{i\in I})$ to $(Y', \{g_i\}_{i\in I})$ iff the diagram in C

$$\begin{tikzcd}
Y \arrow[d, "\alpha"'] \arrow[dr, "f_j"] & \\
Y' \arrow[r, "g_j"'] & X_j
\end{tikzcd}$$

commutes for all $j \in I$. Further, define composition of morphisms by the composition defined in C. It is then immediately verified that $\mathbf{Prod}_C(S)$ is indeed a category. When C is known, we may omit it in the notation and write $\mathbf{Prod}(S) := \mathbf{Prod}_C(S)$.

Remark. We see here that we explicitely include the source of the indexed morphisms, where at first glance it is not necessary, for given any morphism, we are able to uniquely associate with it its hom-set and therefore its source. However, if we consider the empty product, given an empty set, if we say that X is the source of all the morphisms in $\varnothing$, then X can be any object and therefore is not uniquely determined. If we define the objects of $\mathbf{Prod}(S)$ to be $\{f_i\}_{i\in I}$, such that $f_j : Y \to X_j$ for some $Y \in Ob(C)$, then $\mathbf{Prod}(\varnothing)$ is the empty category.

We shall say that an object $P \in Ob(C)$ is the "product of $\{X_i\}_{i\in I}$" iff P is terminal in the category $\mathbf{Prod}(S)$.

Similarly, we shall say that P is the "coproduct of $\{X_i\}_{i\in I}$" iff P is terminal in the category $\mathbf{Prod}_{C^{\mathrm{op}}}(S)$. That is, it is a product of $\{X_i\}_{i\in I}$ in the opposite category.

A terminal object in C is the object part of the product of zero objects of C, and conversely.

Proof. We can only index the empty set of objects with the empty set; we have the empty map $\varnothing : \varnothing \to \varnothing$. Then a set of morphisms indexed by $\varnothing$ is empty, and therefore we see that an object of $\mathbf{Prod}(\varnothing)$ is always the form of $(Y, \varnothing)$, such that $Y \in Ob(C)$. Now suppose $\alpha : Y \to Y'$. We show that $F(\alpha) : (Y, A) \to (Y', B)$, denote A and B as the empty map. Suppose $i \in I = \varnothing$. Then this is a contradiction so we conclude $\alpha : Y \to Y'$ and $A(i) \circ \alpha = B(i)$. Conversely, if $F(\alpha) : (Y, A) \to (Y', B)$, then by definition, $\alpha : Y \to Y'$. Therefore, we conclude that $\mathbf{Prod}(\varnothing)$ is isomorphic to C.

So we see that $(Y, \varnothing)$ is terminal in $\mathbf{Prod}(\varnothing)$ iff Y is terminal in C. $\square$

Remark. Consider if we had defined the objects of $\mathbf{Prod}(S)$ to be $\{f_i\}_{i\in I}$, such that $f_j : Y \to X_j$ for some $Y \in Ob(C)$. Then we are not able to say which object in C is the product.

So we are able to generalize the notion of a product by focusing on its properties that we would like it to satisfy rather than trying to focus on an actual object. So a product is not only an object in C, but also its morphisms to $\{X_i\}_i$. The point to be emphasized is that in defining products, we do not actually care which object we chose to treat in our category, but rather what the projection morphisms are.

4.1.1 Examples of Products and Coproducts

We have that the usual product in the category of sets, groups, rings, A-modules, topological spaces, are in fact the product in the categorical sense. In the category of Abelian groups, and A-modules, we have that the direct product is the categorical product, and the direct sum is the categorical coproduct. The tensor product of rings is the categorical coproduct of rings.

For example, in the category **Sets**, or any category whose underlying category is **Sets**, up to isomorphism, the product of an indexed set of sets $\{X_i\}_{i\in I}$ is the set $\prod_{i\in I} X_i$. The coproduct is the disjoint union $\coprod_{i\in I} X_i$. We note that depending on the category at hand, the product and coproduct may or may not exist.

4.2 Equalizers and Coequalizers

Given two maps of (not necessarily $\mathcal{U}$-small) sets, $f, g : A \to B$, denote $\mathrm{Eq}(f, g) = \{x \in A \mid f(x) = g(x)\}$. This set is called the "equalizer" of f and g. We shall give a broad characterization for a general category of this notion.

Given morphisms $f, g : A \to B$ in category C, define the category $\mathbf{Eq}_C(f, g)$ as the category whose objects consist of all morphisms $\alpha : X \to A$ for some object X such that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & A \arrow[r, "f"] & B \arrow[d, "id"] \\ & A \arrow[r, "g"] & B \end{tikzcd}$$

commutes. That is to say, $f \circ \alpha = g \circ \alpha$.

Recall that we say that a diagram commutes iff any two paths starting at the same objects and ending at the same objects in the diagram coincide. Then when we say that the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

commutes, this must mean that $f = g$. On the other hand, when we say that the diagram

$$B \xleftarrow{g} A \xrightarrow{f} B$$

commutes, this does not mean that $f = g$.

Therefore we explicitely do not write

$$\begin{tikzcd} X \arrow[r, "\alpha"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

and strongly admonish anyone who does so.

Given morphisms $\alpha : X \to A$, $\beta : Y \to A$, say that θ is a morphism from α to β iff it is a morphism in C such that $\theta : X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[d, "\theta"] \arrow[dr, "\alpha"] & \\ Y \arrow[r, "\beta"'] & A \end{tikzcd}$$

commutes. We see that identity and associativity of composition are respected and thus $\mathbf{Eq}_C(f, g)$ is indeed a category. We want the equalizer to be the largest object in this category. That is, state that an object in this category is the equalizer of f and g iff it is a terminal object.

Now we note that $f, g : B \to A$ in the category C^{op}. In the context of the category C, we shall say that an object is the "coequalizer" of f and g iff it is terminal in the category $\mathbf{Eq}_{C^{\text{op}}}(f, g)$.

We denote the equalizer of f and g as $\text{Eq}(f, g)$. We easily see that the set $\{x \in A \mid f(x) = g(x)\}$ is the equalizer of f and g in the category of sets. We also in fact know that the coequalizer in **Set** exists. Explicitely, suppose that we have $f, g : X \to X'$. Then define the relation R on X' as follows. For $x', y' \in X'$, we write $x'Ry'$ iff $\exists z \in X : f(z) = x', g(z) = y'$. That is, $R := \{(f(z), g(z)) \mid z \in X\}$. Then denote $\sim$ as the smallest equivalence that contains R. Denote $\zeta := X'/\sim$ as the quotient and $h : x' \mapsto [x']$.

We show that $h : X' \to \zeta$ is the coequalizer. We see that we have the commutative diagram in $\mathbf{Set}^{\mathrm{op}}$

$$\begin{tikzcd}
\zeta \arrow[r, "h"] \arrow[dr, "h"'] & X' \arrow[r, "f"] & X \arrow[d, "id"] \\
& X' \arrow[r, "g"] & X
\end{tikzcd}$$

That is, $h \circ f = h \circ g$ in $\mathbf{Set}$. Now suppose we have a map $\alpha : X' \to Y$ such that $\alpha \circ f = \alpha \circ g$. Then denote $R' := \{(x', y') \mid \alpha(x') = \alpha(y')\}$, and denote $\sim'$ as the smallest equivalence containing R'. Then clearly $R \subset R'$ and hence $\sim \subset \sim'$. Denote $\zeta' := X'/\sim'$. Put $\mu : X' \to \zeta'$ by $\mu : x' \mapsto [x']$. Then the diagrams

$$\begin{tikzcd}
& \zeta \arrow[d] \\
X' \arrow[ur, "h"] \arrow[r, "\mu"] \arrow[dr, "\alpha"'] & \zeta' \arrow[d] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& {[x']_{\sim}} \arrow[d, mapsto] \\
x' \arrow[ur, mapsto, "h"] \arrow[r, mapsto, "\mu"] \arrow[dr, mapsto, "\alpha"'] & {[x']_{\sim'}} \arrow[d, mapsto] \\
& \alpha(x')
\end{tikzcd}$$

commute and therefore we have a map γ such that $\alpha = \gamma \circ h$. For uniqueness of γ, simply observe that if $\alpha(x') = \gamma'(h(x'))$, we must have $\gamma'([x']_{\sim}) = \alpha(x')$, which is the definition of γ. Therefore h is the coequalizer.

Proposition 66. *A terminal object in* $\mathbf{Eq}_C(f, g)$ *is a monomorphism (in* C*)*

Proof. Suppose we have $\alpha, \beta : Y \to X$, such that $\mathrm{eq} \circ \alpha = \mathrm{eq} \circ \beta$. Then the diagram

$$\begin{tikzcd}
Y \arrow[r, "\alpha"] \arrow[d, "id"] & E \arrow[r, "\mathrm{eq}"] \arrow[d, "id"] & A \arrow[r, "f"] & B \arrow[d, "id"'] \\
Y \arrow[r, "\beta"'] & E \arrow[r, "\mathrm{eq}"] & A \arrow[r, "g"] & B
\end{tikzcd}$$

commutes, which obtains $\alpha = \beta$. $\square$

We know that kernels and cokernels are important in algebra, as they appear in group theory, ring theory, module theory and so on. In this section we shall treat a generalized notion of kernels and cokernels.

4.2.1 Existence of Equalizers in A-Modules

Proposition 67. *Equalizers exist in the category* $\mathbf{Mod}(A)$

Proof. Given module homomorphisms $f, g : M \to N$, put $E := \{x \in M \mid f(x) = g(x)\}$. We then have that E is an A-module and that the inclusion $inc : E \to M$ gives a homomorphism that is the equalizer. $\square$

4.2.2 Existence of Coequalizers in A-Modules

Proposition 68. *Coequalizers exist in the category* $\mathbf{Ab}$

Proof. Suppose we have homomorphisms $f, g : G \to H$. Define $S := \{f(x) - g(x) \mid x \in G\}$. Define $N := H/ < S >$, and $\mu : H \to N$ by the canonical homomorphism. Then we see that it is an object of $\mathbf{Eq}_{\mathbf{Ab}^{op}}(f, g)$, and quickly verify its universality. $\square$

Proposition 69. *For ring A, coequalizers exist in the category* $\mathbf{Mod}(A)$

Proof. Suppose we have homomorphisms $f, g : M' \to M$. Define, similarly, $S := \{f(x) - g(x) \mid x \in M\}$ and $N := M/ < S >$. $\square$

4.3 Kernels and Cokernels

Given category C with a null-map, where every two elements $X, Y \in Ob(C)$, there exists a zero morphism $0_{X,Y} : X \to Y$, and given morphism $f : X \to Y$, consider the category $\mathbf{Ker}(f)$, which we define as follows.

Take objects of $\mathbf{Ker}(f)$ as the morphisms $\alpha : K \to X$ such that the diagram

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ K \arrow[u, "\alpha"] \arrow[r, "0_{K,Y}"'] & Y \end{tikzcd}$$

commutes. We want the kernel to be the most general object (in a sense, the biggest object K along with its associated morphism to X) that satsifies this property. Suppose we have two objects $\alpha : K \to X$ and $\beta : M \to X$. Then state that θ is a morphism from α to β iff the diagram

$$\begin{tikzcd} & X \\ M \arrow[ur, "\beta"] \arrow[r, "\theta"'] & K \arrow[u, "\alpha"'] \end{tikzcd}$$

commutes. The composition is then simply the composition of the morphisms in C. Then indeed, if $\alpha : K \to X$, and $\beta : M \to X$, then . We see that conditions of identity and associativity of composition is clearly obeyed. Therefore **Ker**(f) is indeed a category.

We shall say that an object is the "kernel of f" iff it is terminal in **Ker**(f).

4.4 Pullback and Pushforward

Suppose we have two functions f, g which map from two different sets to the same set. That is, we have $f : A \to X$, $g : B \to Y$. The set

$$S := \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

is called the "fiber product" (fibered product), or the "pullback" of f and g. We wish to generalize this notion in an arbitrary category.

Example 70. Given two morphisms $f : A \to X$, $g : B \to X$ in some category C, we shall denote the **Pullback**(f, g) as the category which has the pair (L, α, β) such that the diagram

$$\begin{tikzcd}
& L \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

commutes. Given two objects (L, α, β) and (L', α', β'), we shall say that $\mu : L \to L'$ is a morphism from (L, α, β) to (L', α', β') iff the diagram

$$\begin{tikzcd}
& L \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, "\mu"] & \\
A & & B \\
& X \arrow[ul, "\alpha'"] \arrow[ur, "\beta'"'] &
\end{tikzcd}$$

commutes. Then **Pullback**(f, g) is indeed a category, and the terminal object in it is called the "pullback" of f and g.

Proposition 71. *The kernel is a special case of the equalizer. Explicitely, if C is a category with a null-map, and $f : X \to Y$ is a morphism, then the kernel of f is the equalizer of $(f, 0_{X,Y})$ and conversely.*

Proof. We note that the three diagrams

$$\begin{tikzcd} & & Y \\ K \arrow[r, "\alpha"] & X \arrow[ur, "f"] \arrow[dr, "0_{X,Y}"'] & \\ & & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \\ K \arrow[ur, "0_{K,Y}"] \arrow[r, "\alpha"] \arrow[dr, "0_{K,Y}"'] & X \arrow[u, "f"'] \arrow[d, "0_{X,Y}"] \\ & Y \end{tikzcd}$$

$$\begin{tikzcd} & Y \\ K \arrow[ur, "0_{K,Y}"] \arrow[r, "\alpha"] & X \arrow[u, "f"'] \end{tikzcd}$$

commute iff one of them commute, due to properties of the null-map. Therefore the objects $\mathbf{Ker}(f)$ are equal to the objects of $\mathbf{Eq}_C(f, 0_{X,Y})$. Further, we see that the collection of morphisms of two objects in $\mathbf{Ker}(f)$ are equal to those of $\mathbf{Eq}_C(f, 0_{X,Y})$. We also see that the laws of composition and therefore the categories are exactly identical. Therefore a terminal object in $\mathbf{Ker}(f)$ is a terminal object of $\mathbf{Eq}_C(f, 0_{X,Y})$, and conversely. □

4.5 Limits and Colimits

Example 72. As we shall see, limits and colimits can be taken as a way to generalize the notion of products and coproducts. This generalization is actually more encompassing than that which meets the eye.

Given category C, and category I, and functor $\alpha : I \to C$ in $Fct(I, C)$, we shall define the "limit" and "colimit" of α. For reasons that will be made clear later, α is called a "diagram". Again one needs to be wary that we need to be able to work with the case where I is the empty category.

Define the category $\mathbf{Lim}(\alpha, I, C)$, which may be called the "category of all cones of/to/on α", as follows. Given functor $\alpha : I \to C$, we note that α can be viewed as a way to index objects of C by the objects of I. Say that a pair, $(X, \{f_i\}_{i \in Ob(I)})$ is an object of $\mathbf{Lim}(\alpha, I, C)$ iff we have

$$f_j : X \longrightarrow \alpha(j)$$

in C for all j, and for all $s \in \mathrm{Hom}_I(j,k)$ the commutativity of the diagram

$$\begin{tikzcd} & \alpha(j) \arrow[d, "\alpha(s)"] \\ X \arrow[ur, "f_j"] \arrow[r, "f_k"'] & \alpha(k) \end{tikzcd}$$

Again, we shall call X the "object part" and $\{f_i\}_i$ the "morphism part" or the "cone" of $(X, \{f_i\}_{i \in Ob(I)})$. When $(L, \{\lambda_i\}_i)$ is the limit cone, we shall call L the "limit object of α" and $\{\lambda_i\}_i$ the "projection morphisms of α".

The reason behind the usage of the word "cone," again, shall be made clear later.

Given two objects, $A = (X, \{f_i\}_{i \in Ob(I)})$ and $B = (Y, \{g_i\}_{i \in Ob(I)})$, we shall say that $\mu : X \to Y$ in C is a morphism from A to B iff the diagram

$$\begin{tikzcd} & \alpha(j) \\ X \arrow[ur, "f_j"] \arrow[r, "\mu"'] & Y \arrow[u, "g_j"'] \end{tikzcd}$$

commutes for all j; that is, $f_j = g_j \circ \mu$. Then we see that identity exists and associativity is respected. We shall say that object A is the limit of α iff it is terminal in the category $\mathbf{Lim}(\alpha, I, C)$. In short, a limit is a terminal cone of α. One may call the object part of the limit the "limit of α" but to be precise, the object itself is insufficient information for it to be called a limit. One may also call a terminal cone a "limit cone".

Now we see that we also have $\alpha^{\mathrm{op}} : I^{\mathrm{op}} \to C^{\mathrm{op}}$. Then we define the colimit of α to be the terminal object in the category $\mathbf{Lim}(\alpha^{\mathrm{op}}, I^{\mathrm{op}}, C^{\mathrm{op}})$. An object of $\mathbf{Lim}(\alpha^{\mathrm{op}}, I^{\mathrm{op}}, C^{\mathrm{op}})$ may be called the "cocone of/to/on α". That is, the colimit of functor α is the limit of its dual cofunctor. In shorthand, one may write colim $\alpha = \lim \alpha^{\mathrm{op}}$.

4.6 Various Universal Objects are Limits or Colimits

In particular, the two previous examples, products and equalizers, are limits. Their dual notions are colimits.

Proposition 73. *The product of objects is a limit, and the coproduct of objects is a colimit.*

Proof. For suppose C is a category, and $S = \{X_i\}_{i \in I}$ is an indexed collection of objects of C, where I is an arbitrary set. Then we are able to define the discrete category of I, which we shall denote $\mathbf{Dis}(I)$. Then denote $\alpha : \mathbf{Dis}(I) \to C$ as the functor which associates

$$i \mapsto X_i$$

and obviously the identity to the identity. Then $\mathbf{Lim}(\alpha, I, C)$ is exactly $\mathbf{Prod}_C(S)$, and therefore a product of $\{X_i\}_i$ is a limit of α and conversely. Further, $\mathbf{Lim}(\alpha^{\mathrm{op}}, I^{\mathrm{op}}, C^{\mathrm{op}})$ is exactly $\mathbf{Prod}_{C^{\mathrm{op}}}(S)$ and therefore coproduct of $\{X_i\}_i$ is a colimit of α and conversely. $\square$

Proposition 74. *Given category C and morphisms $f, g : i \to j$, the equalizer of f and g is a limit, and the coequalizer is a colimit.*

For denote I as the category with precisely two objects i and j, with two morphisms $f, g : i \to j$ along with identities (We may view I as a subcategory of C itself). Denote $\alpha : I \to C$ as the inclusion functor. Then we will show that there is an obvious isomorphism of $\mathbf{Lim}(\alpha, I, C)$ with $\mathbf{Eq}_C(f, g)$.

For we note that the objects in $\mathbf{Lim}(\alpha, I, C)$ are pairs $(Y, \{t_i, t_j\})$ such that we have the commutativity of the diagrams

$$\begin{tikzcd} & Y \arrow[dl, "t_i"'] \arrow[dr, "t_j"] & \\ i \arrow[rr, "f"] & & j \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "t_i"'] \arrow[dr, "t_j"] & \\ i \arrow[rr, "g"] & & j \end{tikzcd}$$

from which we have $f \circ t_i = t_j = g \circ t_i$. Further, a morphism μ from $(X, \{t_i, t_j\})$ to $(Y, \{\tau_i, \tau_j\})$ is one such that we have $\mu : X \to Y$ and

$$\begin{tikzcd} & \alpha(i) \\ X \arrow[ur, "t_i"] \arrow[r, "\mu"'] & Y \arrow[u, "\tau_i"'] \end{tikzcd} \qquad \begin{tikzcd} & \alpha(j) \\ X \arrow[ur, "t_j"] \arrow[r, "\mu"'] & Y \arrow[u, "\tau_j"'] \end{tikzcd}$$

commute. Then we see that if we associate $(Y, \{t_i, t_j\})$ to $t_i \in Ob(\mathbf{Eq}_C(f, g))$, and associate μ to μ in $\mathrm{Hom}(\mathbf{Eq}_C(f, g))$, we obtain an isomorphism of categories. For indeed, we clearly see that the associations are injective. Further, if $\alpha : Y \to i$ is an object of $\mathbf{Eq}_C(f, g)$, then we can associate it to $(Y, \alpha, f \circ \alpha)$. So the maps are also surjective.

Recall that the coequalizer of $f, g : i \to j$ is the terminal object in $\mathbf{Eq}_{\mathbf{Set}^{\mathrm{op}}}(f, g)$, that is, it is the equalizer of f and g viewed in the opposite set category. This that this means that it is the limit of some functor $\alpha : I \to C$, which means that it is a colimit of some functor.

Proposition 75. *The terminal object of any category is a limit (Limit of empty diagram is terminal object)*

Given category C, a terminal object is associated with a limit, and an initial object is associated with a colimit. For put I as the empty category, and $\alpha : I \to C$. Then the collection of objects of $\mathbf{Lim}(\alpha, I, C)$ coincides with that of $\mathbf{Prod}_C(\varnothing)$. Further, the hom-sets and the laws of composition of the two categories coincide, and therefore the categories are the same. Then the object part of the limit is terminal and the object part of the colimit is initial in C.

4.7 Intuition of the word "Diagram" and "Cone" in the definition of a limit

Now that we have seen examples of particular cases of the limit, such as the product and the equalizer, we shall consider what it means. Let us restrict our attention to a category I with a finite set of objects. Then if we write out all the morphisms from each object to every other object, we might get, for example, something like

$$\begin{tikzcd}
& \bullet \arrow[dl] \arrow[dr] & \\
\bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet
\end{tikzcd}$$

where we omit from the diagram the identity morphisms. Applying the functor and placing an object above the diagram, a cone is so called because we have the shape

$$\begin{tikzcd}
& Y \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& \bullet \arrow[dl] \arrow[dr] & \\
\bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet
\end{tikzcd}$$

if we imagine the image of I in the x, y dimension and Y in the z dimension. The condition required is that we need that each triangular diagram with one vertex at Y needs to be commutative (the identity makes it obviously commutative and that is why we did not need to include it in the diagram).

We note that the definition of a limit is not concerned with the nature of the objects or morphisms in I, but is only concerned with how the morphisms relate to one another. Therefore I is only necessary for its "shape" or "diagram".

Remark. In particular, when C is a locally small category, and I is small, and $\alpha : I \to C$, we can consider α as an morphism in the category **LSmall**. Then since α belongs to a unique hom-set, simply specifying α also specifies I as its source and C as its target, so we may simply write $\mathbf{Lim}(\alpha)$ to denote $\mathbf{Lim}(\alpha, I, C)$. Since we have $\alpha^{\text{op}} : I^{\text{op}} \to C^{\text{op}}$ in **LSmall**, we can write $\mathbf{Lim}(\alpha^{\text{op}})$ to denote $\mathbf{Lim}(\alpha^{\text{op}}, I^{\text{op}}, C^{\text{op}})$

5 Functorial Definition of Universal Objects

5.1 Functorial Definition of Products and Coproducts

Suppose C is a locally small category. Although we already characterized the products with their universal property, we shall see that functors can also obtain the same definition. Suppose we have set of objects $\{X_i\}_{i\in I} \subset Ob(C)$, indexed by a small set I.

Denote $\prod_{i\in I} \text{Hom}_C(Y, X_i)$ as the cartesian product of the Hom sets, and suppose $\alpha : Y \to Y'$ in C^{op}. Given $\{f_i\}_{i\in I} \in \prod_{i\in I} \text{Hom}_C(Y, X_i)$, denote $\prod_{i\in I} \bullet \circ \alpha$ as the map which takes $\{f_i\}_{i\in I}$ to $\{f_i \circ \alpha\}_{i\in I} \in \prod_{i\in I} \text{Hom}_C(Y', X_i)$.

Given object Y, denote $FY := \{(Y, t) \mid t \in \prod_{i\in I} \text{Hom}_C(Y, X_i)\}$

For lack of a good symbol, denote F as the functor from C^{op} to **Set**, which takes

$$Y \mapsto FY$$

$$\alpha \mapsto \prod_{i\in I} \text{Hom}_C(\alpha, X_i) = \prod_{i\in I} \bullet \circ \alpha$$

We note that identity and composition are preseved and thus F is indeed a functor.

Now we also note that $\{X_i\}_i$ is also a collection of objects of C^{op}, which is also locally small, and therefore denote $\overline{F} : C \to \mathbf{Set}$ as the functor defined in the same way as F.

Definition 76. If $\theta : F \to \text{Hom}_C(\bullet, P)$ is an isomorphism of functors, then we shall say that the pair (P, θ) is a "product" of $\{X_i\}_{i\in I}$ in the category C, and conversely. Note that the product is not necessarily uniquely defined.

An object P of category C is colloquially/informally called the "product of $\{X_i\}_{i\in I}$" iff it is a representation of the functor F. We note that the object P itself is usually not sufficient information when defining the "product".

Proposition 77. *The above definition yeilds the definition of the product via* $\mathbf{Prod}(\{X_i\}_{i\in I})$. *An object representing* F *is the object part of a terminal object in* $\mathbf{Prod}(\{X_i\}_{i\in I})$, *and conversely.*

In particular, if we have explicit isomorphism $\theta : F \to Hom_C(\bullet, P)$, *then* $(\theta_P)^{-1}(id_P)$ *is the terminal object in the category* $\mathbf{Prod}(\{X_i\}_{i\in I})$. *If* $(P, \{\pi_i\}_i)$ *is terminal in* $\mathbf{Prod}(\{X_i\}_{i\in I})$, *then we can define* θ *by an explicit map defined below, which makes* P *the representative of* F.

Proof. We want to prove that $(\theta_P)^{-1}(id_P)$ is the terminal object in the category $\mathbf{Prod}(\{X_i\}_{i\in I})$. Indeed, we observe that it is indeed an indexed collection morphisms from P to X_i along with the object P. Denote it as $(P, \{\pi_i\}_i)$. Suppose Y is an object, and suppose $(Y, \{f_i\}) \in F(Y)$. Then $\theta(Y)$ takes $\{f_i\}_{i\in I}$ to some $g : Y \to P$. We obtain the commutativity of

$$\begin{tikzcd} F(P) \arrow[r, "\theta_P"] \arrow[d, "\prod \circ g"'] & \text{Hom}(P,P) \arrow[d, "\circ g"] \\ F(Y) \arrow[r, "\theta_Y"] & \text{Hom}(Y,P) \end{tikzcd}$$

$$\begin{tikzcd} (P, \{\pi_i\}_i) \arrow[r, mapsto, "\theta_P"] \arrow[d, mapsto, "\prod \circ g"] & id_P \arrow[d, mapsto, "\circ g"] \\ (Y, \{f_i\}_i) \arrow[r, mapsto, "\theta_Y"] & g \end{tikzcd}$$

which means that $\{\pi_i \circ g\}_i = \{f_i\}_i$, which shows that the diagram

$$\begin{tikzcd} Y \arrow[d, "g"] \arrow[dr, "f_j"] & \\ P \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

commutes for all j. Further, the morphism g is unique, for if g' is a morphism such that $(Y, \{\pi_i \circ g'\}_i) = (Y, \{f_i\}_i)$, then applying θ, we get $id_P \circ g' = g$. This obtains that $(P, \{\pi_i\}_i)$ is in fact the universal object that is desired.

Remark. Note that in the case of the empty product, we are able to associate an object $(P, \varnothing)$. If we did not associate an element with the indexed set of morphisms, we would not be able to uniquely define the empty product (for any element Y satisfies the condution that it is the source of all morphisms of the empty set).

Conversely, suppose that $(P, \{\pi_i\}_{i\in I})$ is the terminal object in $\mathbf{Prod}(\{X_i\}_{i\in I})$. We shall show that such P is a representation of F, that is, $F \approx \mathrm{Hom}(\bullet, P)$. It suffices to define such a morphism.

For suppose $Y \in Ob(C)$, and suppose that $(Y, \{f_i\}_i) \in F(Y)$. Take unique g such that the diagram

$$\begin{tikzcd} Y \arrow[d, "g"'] \arrow[dr, "f_j"] & \\ P \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

commutes for all $i \in I$. Define θ_Y as the map which takes $\{f_i\}$ to such g. We immediately see the commutativity of

$$\begin{tikzcd} F(Y) \arrow[r, "\theta_Y"] \arrow[d, "\prod \circ\alpha"'] & \mathrm{Hom}(Y, P) \arrow[d, "\circ\alpha"] \\ F(Y') \arrow[r, "\theta_{Y'}"] & \mathrm{Hom}(Y', P) \end{tikzcd}$$

$$\begin{tikzcd} (Y, \{f_i\}_i) \arrow[r, mapsto, "\theta_Y"] \arrow[d, mapsto, "\prod \circ\alpha"] & g \arrow[d, mapsto, "\circ\alpha"] \\ (Y', \{f_i \circ \alpha\}_i) \arrow[r, mapsto, "\theta_{Y'}"] & g \circ \alpha \end{tikzcd}$$

which proves that P is a representation of F. □

Definition 78. An object P of category C is called the "coproduct of $\{X_i\}_{i\in I}$" iff it is a representation of the functor $\overline{F}$.

Corollary. *If we have isomorphism $\theta : \overline{F} \to Hom_{C^{op}}(\bullet, P)$, and thus $(\theta_P)^{-1}(id_P)$ is the terminal object in the category $\mathbf{Prod}_{C^{op}}(\{X_i\}_{i\in I})$. Conversely, if*

$(P, \{inc_i\}_{i\in I})$ is the terminal object in $\mathbf{Prod}_{C^{op}}(\{X_i\}_{i\in I})$, then we have isomorphism $\overline{F} \approx Hom_{C^{op}}(\bullet, P)$, which is to say that P is the coproduct of $\{X_i\}_{i\in I}$.

Proof. Self evident. $\square$

5.2 Functorial Definition of Equalizers and Coequalizers

Recall that given two maps of (not necessarily $\mathcal{U}$-small) sets, $f, g : A \to B$, we denote $\mathrm{Eq}(f, g) = \{x \in A \mid f(x) = g(x)\}$.

Let C denote a locally small category, and suppose we have morphisms $f, g : X \to X'$ in C. We shall define a functor $F : C^{\mathrm{op}} \to \mathbf{Set}$. Given $Y \in Ob(C^{\mathrm{op}})$, denote $F(Y) := \mathrm{Eq}(\mathrm{Hom}_C(Y, f), \mathrm{Hom}_C(Y, g))$. Further, given morphism $\alpha : Y' \to Y$ in C^{op}, we define the map $F(\alpha) : F(Y') \to F(Y)$ by the association

$$F(\alpha) : t \mapsto t \circ \alpha$$

Then indeed, we see that if $t : Y' \to X$ is in $\mathrm{Eq}(\mathrm{Hom}_C(Y', f), \mathrm{Hom}_C(Y', g))$, that is, it satisfies $f \circ t = g \circ t$, then we have $f \circ (t \circ \alpha) = g \circ (t \circ \alpha)$. Further, we see that F respects identity and composition.

Now we have that $f, g : X' \to X$ in the category C^{op}, which is also a locally small category. We therefore have a functor $\overline{F} : C \to \mathbf{Set}$ defined by (f, g).

Definition 79. An object of C is said to be an "equalizer" of (f, g) iff it is a representation of the functor F.

Proposition 80. *The above definition yeilds the definition of an equalizer via $\mathbf{Eq}_C(f, g)$. An object representing F is the source of some terminal object in $\mathbf{Eq}_C(f, g)$, and conversely.*

In particular, if we have explicit isomorphism $\theta : F \to Hom_C(\bullet, E)$, then $(\theta_E)^{-1}(id_E)$ is the terminal object in the category $\mathbf{Eq}_C(f, g)$. If eq is terminal in $\mathbf{Eq}_C(f, g)$, then its source is the representation of F.

Proof. Suppose Y is some object, and $t \in \mathrm{Eq}(\mathrm{Hom}_C(Y, f), \mathrm{Hom}_C(Y, g))$. Then θ_Y takes t to some $\alpha : Y \to E$, a morphism in C. Then denote the inverse image of id_E as eq. We have the commutativity of

$$\begin{tikzcd} F(E) \arrow[r, "\theta_E"] \arrow[d, "\circ\alpha"] & \mathrm{Hom}(E, E) \arrow[d, "\circ\alpha"] \\ F(Y) \arrow[r, "\theta_Y"] & \mathrm{Hom}(Y, E) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{eq} \arrow[r, mapsto, "\theta_E"] \arrow[d, mapsto, "\circ\alpha"] & id_E \arrow[d, mapsto, "\circ\alpha"] \\ t \arrow[r, mapsto, "\theta_Y"] & \alpha \end{tikzcd}$$

This shows that the diagram

$$\begin{tikzcd} Y \arrow[d, "\alpha"] \arrow[dr, "t"] & \\ E \arrow[r, "\mathrm{eq}"'] & X \end{tikzcd}$$

commutes. To show uniqueness of α, suppose that we have α' also satisfies $\mathrm{eq} \circ \alpha' = t$. Then applying θ, we obtain $id_E \circ \alpha' = \alpha$. More explicitely, we have that the diagram

$$\begin{tikzcd} \mathrm{eq} \arrow[r, mapsto, "\theta_E"] \arrow[d, mapsto, "\circ\alpha'"] & id_E \arrow[d, mapsto, "\circ\alpha'"] \\ \mathrm{eq} \circ \alpha' \arrow[r, mapsto, "\theta_Y"] & \alpha' \end{tikzcd}$$

commutes, and hence $\theta_Y(t) = \theta_Y(\mathrm{eq} \circ \alpha')$, obtaining $\alpha = \alpha'$.

Conversely, suppose that eq is terminal in the category $\mathbf{Eq}_C(f, g)$. Denote E as the source of the morphism eq. We show that $F \approx \mathrm{Hom}(\bullet, P)$.

For suppose $Y \in Ob(C)$, and suppose that $t \in F(Y)$. Take unique μ such that the diagram

$$\begin{tikzcd} Y \arrow[d, "\mu"'] \arrow[dr, "t"] & \\ E \arrow[r, "\mathrm{eq}"'] & X \end{tikzcd}$$

commutes for all $i \in I$. Define θ_Y as the map which takes t to such μ. We immediately see the commutativity of

$$\begin{tikzcd} F(Y) \arrow[r, "\theta_Y"] \arrow[d, "\circ\alpha"] & \mathrm{Hom}(Y, P) \arrow[d, "\circ\alpha"] \\ F(Y') \arrow[r, "\theta_{Y'}"] & \mathrm{Hom}(Y', P) \end{tikzcd}$$

$$\begin{tikzcd} t \arrow[r, mapsto, "\theta_Y"] \arrow[d, mapsto, "\circ\alpha"] & \mu \arrow[d, mapsto, "\circ\alpha"] \\ t \circ \alpha \arrow[r, mapsto, "\theta_{Y'}"] & \mu \circ \alpha \end{tikzcd}$$

which proves that E is a representation of F. $\square$

Definition 81. An object of C is said to be an "coequalizer" of (f, g) iff it is a representation of the functor $\overline{F}$.

Corollary. *The above definition yeilds the definition of an equalizer via* $\mathbf{Eq}_{C^{op}}(f, g)$*. In particular, if we have explicit isomorphism* $\theta : \overline{F} \to Hom_{C^{op}}(\bullet, E)$*, then* $(\theta_E)^{-1}(id_E)$ *is the terminal object in the category* $\mathbf{Eq}_{C^{op}}(f, g)$*. If eq is terminal in* $\mathbf{Eq}_{C^{op}}(f, g)$*, then its source is the representation of* $\overline{F}$*.*

Proof. Self evident. $\square$

Recall that the kernel is a special case of the equalizer.

Definition 82. Given a null-map category C, and morphism $f : X \to X'$ of C, an object $E \in Ob(C)$, we say that it is the kernel of f iff it is the equalizer of f and $0_{X,X'}$.

It is easily seen that the above definition yeilds the definition via the universal property of $\mathbf{Ker}(f)$. In particular, if an object is the kernel of f, then it is the source of a terminal object of $\mathbf{Ker}(f) = \mathbf{Eq}_{C^{op}}(f, 0_{X,X'})$, and conversely.

5.3 Functorial Definition of Limits and Colimits

Let I be a small category. Given functor $\alpha : I \to \mathbf{Set}$, the cartesian product $\prod_i \alpha(i)$ is small. Then the limit of α is defined as the small set:

$$\varprojlim \alpha := \left\{ \{x_i\}_i \in \prod_i \alpha(i) \mid \alpha(s)\,(x_j) = x_k \text{ for all } s \in \mathrm{Hom}_I(j,k) \right\}$$

Remark. Note here that the jth projective map, π_j which takes $\{x_i\}_i$ to x_j, is the desired λ_i, which formally makes $(\varprojlim \alpha, \{\lambda_i\}_i)$ the set limit in our old definition.

We see that if I is the empty category, then $\varprojlim \alpha$ is the singleton that contains the empty set, which is indeed terminal in **Set**. We recall that we have the covariant functor $\mathrm{Hom}_{\mathbf{Set}}(X, \bullet) : \mathbf{Set} \to \mathbf{Set}$, and denote $\mathrm{Hom}_{\mathbf{Set}}(X, \alpha\bullet)$ as the composition $\mathrm{Hom}_{\mathbf{Set}}(X, \bullet) \circ \alpha$.

Now that we have defined the limit for **Sets**, we shall use this to define limits and colimits in general.

Let C denote a locally small category, and I denote a small category. Suppose

$$X \in Ob(C)$$

$$\alpha : I \to C; \text{ that is, } \alpha : I^{\mathrm{op}} \to C^{\mathrm{op}}$$

Recalling that since C is locally small, we have covariant functors

$$\mathrm{Hom}_C(X, \bullet) : C \to \mathbf{Set}$$

$$\mathrm{Hom}_C(\bullet, X) : C^{\mathrm{op}} \to \mathbf{Set}$$

so we have their compositions

$$\mathrm{Hom}_C(X, \alpha\bullet) : I \to \mathbf{Set}$$

$$\mathrm{Hom}_C(\alpha\bullet, X) : I^{\mathrm{op}} \to \mathbf{Set}$$

We also recall, from the discussion of the composition of bifunctors with functors, that

$$\mathrm{Hom}_C(\bullet, \alpha\bullet) : C^{\mathrm{op}} \times I \to \mathbf{Set}$$

$$\mathrm{Hom}_C(\alpha\bullet, \bullet) : I^{\mathrm{op}} \times C \to \mathbf{Set}$$

are bifunctors.

Continuing to let I be small, denote $F : C^{\mathrm{op}} \to Fct(I, \mathbf{Set})$ as the functor which associates $X \in Ob(C^{\mathrm{op}})$ to $\mathrm{Hom}_C(X, \alpha\bullet)$, and associates $t : X \to Y$ in C^{op} to $\mathrm{Hom}_C(t, \alpha\bullet)$, which associates object i to the morphism $\mathrm{Hom}_C(t, \alpha(i)) = \circ t$ in $\mathbf{Set}$. (See the section on the Yoneda Lemma to recall that F does in fact define a functor).

Further, denote $\Lambda : Fct(I, \mathbf{Set}) \to \mathbf{Set}$ as the functor which associates functor $a : I \to \mathbf{Set}$ to the set $\lim_{\longleftarrow} a$, and morphism of funtors $\theta : a \Longrightarrow b$ to the map which associates $\{x_i\}_i \in \lim_{\longleftarrow} a$ to $\{\theta_i(x_i)\}_i$, which, indeed, due to the commutativity of

$$\begin{tikzcd}
x_k \arrow[r, mapsto, "\theta_k"] \arrow[d, mapsto, "a(s)"] & \theta_k(x_k) \arrow[d, mapsto, "b(s)"] \\
x_j \arrow[r, mapsto, "\theta_j"] & \theta_j(x_j)
\end{tikzcd}$$

is in $\lim_{\longleftarrow} b$. We also see that the association respects identity and composition and is confirmed to be a functor.

Finally we define $\Xi : C^{\text{op}} \to \mathbf{Set}$ which associates object X to $\Lambda \circ F(X)$, and morphism t to $\Lambda \circ F(t)$.

Definition 83. For $L \in Ob(C) = Ob(C^{\text{op}})$, and isomorphism of functors $\theta : \mathrm{Hom}_C(\bullet, L) \Longrightarrow \Xi$, we state that a pair (L, θ) is the limit of $\alpha : I \to C$. That is, we have isomorphism

$$\mathrm{Hom}_C\,(X, L) \approx \Xi(X)$$

functorial in X. We may write $\lim \mathrm{Hom}_C(X, \alpha\bullet) = \Xi(X)$. We usually denote such L as $\lim \alpha$.

Denote $G : C \to Fct(I^{\text{op}}, \mathbf{Set})$ as the functor which associates $X \in Ob(C)$ to $\mathrm{Hom}_C(\alpha\bullet, X)$, and associates $t : X \to Y$ in C to $\mathrm{Hom}_C(\alpha\bullet, t)$, which associates object i to the morphism $\mathrm{Hom}_C(\alpha(i), t)$ in $\mathbf{Set}$.

Define $\overline{\overline{\Xi}} : C^{\text{op}} \to \mathbf{Set}$ which associates object X to $\Lambda \circ G(X)$, and morphism t to $\Lambda \circ G(t)$

Definition 84. For $L \in Ob(C)$, and isomorphism of functors $\theta : \mathrm{Hom}_C(L, \bullet) \Longrightarrow \overline{\overline{\Xi}}$, we state that a pair (L, θ) is the colimit of α. That is, we have isomorphism

$$\mathrm{Hom}_C\,(L, X) \approx \overline{\overline{\Xi}}(X)$$

functorial in X. We may write $\lim \mathrm{Hom}_C(\alpha\bullet, X) = \overline{\overline{\Xi}}(X)$. We usually denote such L as $\lim \alpha$.

Proposition 85. *The definition of limits given via functors coincides with those given by universal properties (Given appropriate smallness conditions). An object representing* $\mathrm{Hom}_C(X, \alpha\bullet)$ *is the object part of some terminal object in* $\mathbf{Lim}(\alpha, I, C)$*, and conversely.*

In particular, if we have explicit isomorphism

$$\theta : \lim_I \mathrm{Hom}_C(X, \alpha\bullet) \to \mathrm{Hom}_C\,(X, L)$$

functorial in X*, then* $(\theta_L)^{-1}(id_L)$ *is the terminal object in the category* $\mathbf{Lim}(\alpha, I, C)$*. If* $(L, \{\lambda_i\}_i)$ *is terminal in* $\mathbf{Lim}(\alpha, I, C)$*, then we can define such* θ *functorial in* X *by an explicit map defined below.*

Proof. We want to prove that $(\theta_L)^{-1}(id_L)$ is terminal.

Indeed, we observe that it is a pair $(L, \{\lambda_i\}_i)$ where we have morphisms from L to $\alpha(i)$ such that the diagram

$$\begin{tikzcd} & \alpha(j) \arrow[d, "\alpha(s)"] \\ L \arrow[ur, "\lambda_j"] \arrow[r, "\lambda_k"'] & \alpha(k) \end{tikzcd}$$

commutes for any $s \in \mathrm{Hom}_I(j,k)$, showing that it is an object of $\mathbf{Lim}(\alpha, I, C)$. Suppose $\{f_i\}_i$ is an object of $\mathbf{Lim}(\alpha, I, C)$; take object Y such that we have $\{f_i\}_{i\in I} \in \lim_C \mathrm{Hom}_C(Y, \alpha\bullet)$. For shorthand, denote $\xi := \lim_C \mathrm{Hom}_C(X, \alpha\bullet)$.

Then $\theta(Y)$ takes $\{f_i\}_{i\in I}$ to some $g : Y \to L$. Then $\xi(g) = \Lambda \circ F(g) = \Lambda(\circ g) = \prod \circ g$. We obtain the commutativity of

$$\begin{tikzcd} \xi(L) \arrow[r, "\theta_L"] \arrow[d, "\xi(g)"] & \mathrm{Hom}(L,L) \arrow[d, "\circ g"] \\ \xi(Y) \arrow[r, "\theta_Y"] & \mathrm{Hom}(Y,L) \end{tikzcd}$$

$$\begin{tikzcd} (L, \{\lambda_i\}_i) \arrow[r, mapsto, "\theta_L"] \arrow[d, mapsto, "\prod \circ g"] & id_L \arrow[d, mapsto, "\circ g"] \\ (Y, \{f_i\}_i) \arrow[r, mapsto, "\theta_Y"] & g \end{tikzcd}$$

which means that $\{\lambda_i \circ g\}_i = \{f_i\}_i$, which shows that the diagram

$$\begin{tikzcd} Y \arrow[d, "g"] \arrow[dr, "f_j"] & \\ L \arrow[r, "\lambda_j"'] & X_j \end{tikzcd}$$

commutes for all j. Further, the morphism g is unique, for if g' is a morphism such that $\{\lambda_i \circ g'\}_i = \{f_i\}_i$, then applying θ, we get $id_L \circ g' = g$. This obtains that $(L, \{\lambda_i\}_i)$ is in fact the universal object that is desired.

Conversely, suppose that $(L, \{\lambda_i\}_{i\in I})$ is the terminal object in $\mathbf{Lim}(\alpha, I, C)$. Denote P as the source of the morphisms λ_i. We shall show that such L is a representation of ξ, that is, $\xi \approx \mathrm{Hom}(\bullet, L)$. It suffices to define such a morphism.

For suppose $Y \in Ob(C)$, and suppose that $(Y, \{f_i\}_i) \in \xi(Y)$. Take unique g such that the diagram

$$\begin{tikzcd} Y \arrow[d, "g"] \arrow[dr, "f_j"] & \\ L \arrow[r, "\lambda_j"'] & \alpha(j) \end{tikzcd}$$

commutes for all $j \in I$. Define θ_Y as the map which takes $(Y, \{f_i\}_i)$ to such g. We immediately see the commutativity of

$$\begin{tikzcd} \xi(Y) \arrow[r, "\theta_Y"] \arrow[d, "\prod \circ\alpha"] & \mathrm{Hom}(Y, P) \arrow[d, "\circ\alpha"] \\ \xi(Y') \arrow[r, "\theta_{Y'}"] & \mathrm{Hom}(Y', P) \end{tikzcd}$$

$$\begin{tikzcd} (Y, \{f_i\}_i) \arrow[r, mapsto, "\theta_Y"] \arrow[d, mapsto, "\prod \circ\alpha"] & g \arrow[d, mapsto, "\circ\alpha"] \\ (Y', \{f_i \circ \alpha\}_i) \arrow[r, mapsto, "\theta_{Y'}"] & g \circ \alpha \end{tikzcd}$$

which proves that $\mathrm{Hom}(\bullet, L)$ and ξ are isomorphic in the category $Fct(C^{\mathrm{op}}, \mathbf{Set})$.

□

Corollary. *The above definition yeilds the definition of an colimit via* $\mathbf{Lim}(\alpha, I^{op}, C^{op})$. *In particular, if we have explicit isomorphism* $\theta : \Lambda \circ G \to Hom_{C^{op}}(\bullet, L)$, *then* $(\theta_L)^{-1}(id_L)$ *is the terminal object in the category* $\mathbf{Lim}(\alpha, I^{op}, C^{op})$. *If* L *is terminal in* $\mathbf{Lim}(\alpha, I^{op}, C^{op})$, *then its source is the representation of* $\Lambda \circ G$.

Proof. Self evident. □

Remark. Note that in the above discussion of limits, the isomorphism of functors allows us to say the same thing about both functors if something is true of one of the functors. For example, suppose we have morphism of functors $F \approx G$ in $Fct(C, D)$, and $\alpha : A \to B$ in C. If $F(\alpha)$ is the only morphism from $F(A)$ to $F(B)$, then $G(\alpha)$ is also the only morphism from $G(A)$ to $G(B)$.

Corollary. *The contravariant Yoneda embedding preserves limits of small shapes, and the covariant Yoneda embedding preserves colimits of small shapes.*

Proof. Having $\alpha : I \to C$, and viewing

$$\mathrm{Hom}_C(\bullet, \alpha\bullet) : I \to Fct(C^{\mathrm{op}}, \mathbf{Set})$$

we have that the statement follows immediately from the isomorphism

$$\mathrm{Hom}_C\left(\bullet, \lim_I \alpha\right) \approx \lim_I \mathrm{Hom}_C(\bullet, \alpha\bullet)$$

which is essentially saying

$$y(\lim_I \alpha) \approx \lim_I (y(\alpha))$$

The covariant case is dual. $\square$

This corollary shall be explained in more detail later in the section on limits and colimits.

6 Limits and Colimits

A limit of a functor $\alpha : I \to C$ is called "finite" iff I is a finite category, that is, $Ob(I)$ is finite.

6.1 Existence of Limits and Colimits in the Category of Sets

Proposition 86. *Suppose $\alpha : I \to \mathbf{Set}$, where I is small. Then following isomorphism holds in $\mathbf{Set}$:*

$$\mathrm{Hom}_{\mathbf{Set}}(X, \lim \alpha) \approx \Xi(X) = \lim \mathrm{Hom}_{\mathbf{Set}}(X, \alpha\bullet)$$

that is functorial in X.

Proof. We first note that if I is empty, since $\lim_{\longleftarrow} \alpha = \{\varnothing\}$, we have $\mathrm{Hom}_{\mathbf{Set}}(X, \lim_{\longleftarrow} \alpha)$ is (also) a singleton. We also have $\lim_{\longleftarrow} \mathrm{Hom}_{\mathbf{Set}}(X, \alpha\bullet) = \{\varnothing\}$. We might for a moment wonder whether there is any iffy business where we forgot to specify an object that is the source of the collection of some morphisms, but we specifically defined the object in the limit category itself, so the empty case works fine.

We have that

$$\lim_{\longleftarrow} \mathrm{Hom}_{\mathbf{Set}}(X, \alpha\bullet)$$

$$= \left\{ \{f_i\}_i \in \prod_i \mathrm{Hom}_{\mathbf{Set}}(X, \alpha(i)) \mid \alpha(s) \circ f_j = f_k \text{ for all } s \in \mathrm{Hom}_I(j,k) \right\}$$

So given map $f : X \to \lim_{\longleftarrow} \alpha$, and $i \in Ob(I)$, denote $f_i : X \to \alpha(i)$ the map which associates $x \in X$ to $f(x)_i$. Then if $s \in \mathrm{Hom}_I(j,k)$, we have that $\alpha(s) \circ f_j : X \to \alpha(k)$. By assumption, we have that $f(x) \in \lim_{\longleftarrow} \alpha$, so $\alpha(s)(f(x))_j = f(x)_j$, which is to say that $\alpha(s)(f_j(x)) = f_k(x)$.

Associate f to such $\{f_i\}_i$, which was shown to be the morphism part of $\lim_{\longleftarrow} \mathrm{Hom}_{\mathbf{Set}}(X, \alpha\bullet)$. Clearly the map is injective. For surjectivity, suppose we are given $\{g_i\}_i \in \lim_{\longleftarrow} \mathrm{Hom}_{\mathbf{Set}}(X, \alpha)$. Then define the map $f : X \to \lim_{\longleftarrow} \alpha$ which associates $x \in X$ to the element $\{g_i(x)\}_i$. Indeed, since $\alpha(s) \circ g_j = g_k$ for all $s \in \mathrm{Hom}_{I^{\mathrm{op}}}(j,k)$, we have that $\alpha(s)(g_j(x)) = g_k(x)$ for all $s \in \mathrm{Hom}_{I^{\mathrm{op}}}(j,k)$, so $f \in \mathrm{Hom}_{\mathbf{Set}}\left(X, \lim_{\longleftarrow} \alpha\right)$.

It is then obvious that $\{f_i\}_i = \{g_i\}_i$, which shows that the association is bijective.

Now suppose $t : X \to Y$ in $\mathbf{Set}^{\mathrm{op}}$. We have the diagram

$$\begin{tikzcd}
\mathrm{Hom}\left(X, \lim_{\longleftarrow} \alpha\right) \arrow[r, "\sim"] \arrow[d, "\circ t"] & \lim_{\longleftarrow} \mathrm{Hom}(X, \alpha\bullet) \arrow[d, "\Lambda(\mathrm{Hom}(t,\beta\bullet))"] \\
\mathrm{Hom}\left(Y, \lim_{\longleftarrow} \alpha\right) \arrow[r, "\sim"] & \lim_{\longleftarrow} \mathrm{Hom}(Y, \alpha\bullet)
\end{tikzcd}$$

Where $\Lambda(\mathrm{Hom}(t, \alpha\bullet)) = \prod \circ t$ is the map which takes $\{f_i\}_i$ to $\{\theta_i(f_i)\}_i = \{f_i \circ t\}_i$.

If $f \in \mathrm{Hom}\left(X, \lim_{\longleftarrow} \alpha\right)$, then going the upper path we simply get $\{f_i \circ t\}_i$, where $f_i(x) := f(x)_i$. So for each i, and $y \in Y$, we evaluate $f(t(y))_i$

Going the lower path, we get $\{(f \circ t)_i\}_i$, where $(f \circ t)_i(y) = (f \circ t(y))_i = f(t(y))_i$. Therefore the diagram commutes. $\square$

Corollary. *In the category of sets, the limit of any small shape exists.*

Proposition 87. *Suppose $\alpha : I \to \mathbf{Set}$, where I is small. Then the colimit of α exists.*

Proof. Note that the limit of $\alpha : I \to \mathbf{Set}$ is a subset of the set product of $\{\alpha(i)\}_i$. That is, we simply needed to impose conditions on the set product to obtain the limit in general. Similarly we also see that the colimit is very much like the coproduct, except we need to impose additional conditions.

Recall that the coproduct comes with the inclusion $x \mapsto (x, i)$ for any $i \in Ob(I)$.

Denote L as the set

$$L := \frac{\coprod \alpha(i)}{\sim}$$

where $\sim$ denotes the smallest equivalence containing the relation R defined by

$$(x, i)R(y, j) \iff \exists s : i \to j \text{ such that } y = \alpha(s)(x)$$

We observe that the relation R is reflexive and transitive.

Then given $i \in Ob(I)$, denote $\gamma_i : \alpha(i) \to L$ as the map $\gamma_i : x \mapsto [(x, i)]$. Then the commutativity conditions needed for $(L, \{\gamma_i\}_i)$ to be an object in $\mathbf{Lim}_C(\alpha, I^{\text{op}}, C^{\text{op}})$ is satisfied. Now suppose (Y, f_i) is an arbitrary object. Define $\mu : L \to Y$ as the map which associates $[(x, j)] \mapsto f_j(x)$, which is well defined.

we have the commutativity of

$$\begin{tikzcd}
& \alpha(j) \arrow[dl, "f_j"'] \arrow[d, "\gamma_j"] \\
Y & L \arrow[l, "\mu"]
\end{tikzcd}$$

for all j, and see that by definition, μ must be unique. $\square$

We note here that the equivalence $\sim$ given in the above proof can be explicitely expressed as follows:

$$(x, i) \sim (y, j)$$

$$\iff$$

there exists morphisms and objects $m_t, b_{t,t+1}$

$$\begin{tikzcd}
& \mu_1 \arrow[dl] \arrow[dr] & & \mu_2 \arrow[dl] \arrow[dr] & & \mu_n \arrow[dl] \arrow[dr] & \\
i = a_{0,1} & & a_{1,2} & & \cdots & & j = a_{n,n+1}
\end{tikzcd}$$

$$\text{such that } m_t \in \mu_t \text{ maps to } b_{t-1,t} \in a_{t-1,t} \text{ and } b_{t,t+1} \in a_{t,t+1}$$
$$\text{where } x = b_0, y = b_n$$

It is easily seen that this is indeed an equivalence that contains R. To show that it is the smallest such equivalence, recall (from FOA) that the smallest such equivalence is the set of all elements (x, y) in $\coprod \alpha(i)$ such that there exists a finite chain

$$(x_0, x_1), (x_1, x_2), \cdots, (x_{n-1}, x_n)$$

of elements such that $x_0 = x$, and $x_n = y$, and either $(x_i, x_{i+1}) \in R$ or $(x_{i+1}, x_i) \in R$ for $i = 0, \cdots, n-1$, or such that $x = y$. This corresponds to the set of all elements (x, y) with the associated finite chain

$$(x_0, x_1), (x_1, x_2), \cdots, (x_n, x_{n+1})$$

such that $(x_i, x_{i+1}) \in R$ for even i, and $(x_{i+1}, x_i) \in R$ for odd i, and n is odd (Note that this only holds because R is already reflexive).

When denoting limits of functors that take values in **Set**, it might be more conventional to typset it as

$$\bigsqcup_i \alpha(i)$$

instead of

$$\coprod_i \alpha(i)$$

6.2 General statements regarding Limits and Colimits

6.3 Projective Limits (Inverse Limits) and Inductive Limits (Direct Limits)

Since in literature, one may encounter confusing terminology regarding limits and colimits, we make a few remarks.

The "inverse limit", also called the "projective limit" is a special type of limit. In particular, an ordered pair (L, λ) is said to be a "projective limit", or an "inverse limit" iff there exists a functor F from a partial order category $\mathbf{Poset}(I, \leq)$ to some category C such that (L, λ) is the limit of F.

The "direct limit", also called that "inductive limit" is a special type of colimit. In particular, an ordered pair (L, λ) is said to be a "direct limit", or an "inverse limit" iff there exists a functor F from a partial order category $\mathbf{Poset}(I, \leq)$ to some category C such that (L, λ) is the colimit of F.

6.4 If Index has Initial Object, then Functor has Limit

Suppose we have arbitrary categories I and C, and functor $\alpha : I \to C$. Suppose that x is initial in I. Then the object $\alpha(x)$ is isomorphic to $\lim \alpha$ in C. For given $i \in I$, associate ρ_i as the application of α to the unique morphism from x to i. Then $(\alpha(x), \rho)$ is an object of $\mathbf{Lim}(\alpha, I, C)$. We show that it is terminal. For suppose we have object $(Y, \{f_i\}_i)$ in the limit category. We want the existence of g in the diagram

$$\begin{tikzcd} Y \arrow[d, "g"] \arrow[dr, "f_j"] & \\ \alpha(x) \arrow[r, "\rho_j"'] & \alpha(j) \end{tikzcd}$$

which we can give as $g := f_x$, and because $(Y, \{f_i\}_i)$ is an object in the category of cones, and ρ_j is $\alpha(t)$ of some morphism $t : x \to j$, the diagram commutes.

6.5 Completeness and Cocompleteness (Limit Functor)

For categories C and I, we will say that C "admits limits of shape I" or "is complete with respect to shape I" iff for any functor $\alpha : I \to C$, the limit of α exists. We will say that C "admits colimits of shape I" or "is cocomplete with respect to shape I" iff for any functor $\alpha : I \to C$, the colimit of α exists. The category of sets is complete with respect to any shape I. We can also just say "C admits limits/colimits" or "C is complete" when it admits limits/coimits of any shape.

Analogously, if given any index set I of objects in C, the product of those objects exist in C, we shall say that "C admits products". If given any two morphisms $f, g : A \to B$ in C, the equalizer of f and g exists, we shall say that "C admits equalizers". And so on.

If C is complete with respect to shape I, we can define a functor $\lim : Fct(I, C) \to C$ in the following manner.

When $\theta : \alpha \implies \beta$, and we have limits of α and β, which we denote $(L_\alpha, \lambda_\alpha)$ and (L_β, λ_β), respectively, we note that $(L_\alpha, \{\theta_j \circ \lambda_\alpha(j)\}_j)$ is an element of $\mathbf{Lim}(\beta, I, C)$, due to the commutativity of the diagram

$$\begin{tikzcd}
L_\alpha \arrow[r, "\lambda^\alpha(j)"] \arrow[dr, "\lambda^\alpha(k)"'] & \alpha(j) \arrow[r, "\theta_j"] \arrow[d, "\alpha(s)"] & \beta(j) \arrow[d, "\beta(s)"] \\
 & \alpha(k) \arrow[r, "\theta_k"] & \beta(k)
\end{tikzcd}$$

Then there exists a unique $\mu_{\alpha,\beta} : L_\alpha \to L_\beta$ in $\mathbf{Lim}(\beta, I, C)$ such that

$$\begin{tikzcd}
\lim \alpha \arrow[r, "\lambda_i^\alpha"] \arrow[d, "\mu_{\alpha,\beta}"'] & \alpha(i) \arrow[d, "\theta(i)"] \\
\lim \beta \arrow[r, "\lambda_i^\beta"'] & \beta(i)
\end{tikzcd}$$

commutes, so we define the functor lim by associating

$$\lim : \alpha \mapsto \lim \alpha$$

$$\lim : \theta \mapsto \mu_{\alpha,\beta}$$

and one easily verifies that identity and composition are preserved. One calls this functor the "limit functor".

6.6 Limits in Two Shapes (Limit Over a Product Shape; Double Limits)

Suppose C, D, and $\mathbb{A}$ are categories, and $\mathbb{A}$ admits limits of shape C. If $\alpha : C \times D \to \mathbb{A}$ is a functor, and $A \in D$, then we have

$$\alpha(\bullet, A) : C \to \mathbb{A}$$

is a functor and therefore has a limit, which is an object of $\mathbb{A}$. Suppose $t : A \to B$. Define $F : C \to D$ as the association which takes

$$F : A \mapsto \lim_C(\alpha(\bullet, A))$$

$$F : t \mapsto \lim_C(\alpha(\bullet, t))$$

Then F is a functor, because, recalling from the section on bifunctors, that

$$\Xi^t : D \longrightarrow Fct(C, \mathbb{A})$$

is a functor, F is the composition $\Xi^t \circ \lim$.

Proposition 88. *(Limits in product categories) If $F : D \to Fct(C, \mathbb{A})$ is a functor, and the limit $\lim_{A \in D}(F(A)(X))$ exists for all $X \in Ob(C)$, then the limit of F, which is a functor from C to $\mathbb{A}$, exists, and in particular, is given by*

$$(\lim_D F)(X) = \lim_{A \in D}(F(A)(X))$$

In short, limits are computed pointwise.

Proof. Recall that given object $X \in Ob(C)$, there exists an evaluation functor $E_X : Fct(C, \mathbb{A}) \to \mathbb{A}$. Then for $F : D \to Fct(C, \mathbb{A})$, we have that $E_X \circ F : D \to \mathbb{A}$, and therefore $E_X \circ F$ has a limit in $\mathbb{A}$. Denote such a limit as $(L^{X,F}, \lambda^{X,F})$. This limit is defined by the commutativity of the diagram

$$\begin{tikzcd}
& L^{X,F} \arrow[dl, "\lambda^{X,F}(A)"'] \arrow[dr, "\lambda^{X,F}(B)"] & \\
F(A)(X) \arrow[rr, "F(t)(X)"'] & & F(B)(X)
\end{tikzcd}$$

in $\mathbb{A}$for all $t : A \to B$ in D.

The limit exists for any X, so define a functor $V$from C to $\mathbb{A}$ by mapping

$$V : X \mapsto L^{X,F}$$

for $X \in Ob(C)$. Given $\eta : X \to Y$ in $\mathrm{Hom}(C)$, and $A \in Ob(D)$, we have $(L^{X,F}, \{F(A)(\eta) \circ \lambda^{X,F}(A)\}_A)$ is an object of $\mathbf{Lim}(E_Y \circ F, D, \mathbb{A})$, due to the comutativity of

$$\begin{tikzcd}
& L^{X,F} \arrow[dl, "\lambda^{X,F}(A)"'] \arrow[dr, "\lambda^{X,F}(B)"] & \\
F(A)(X) \arrow[rr, "F(t)(X)"'] \arrow[d, "F(A)(\eta)"'] & & F(B)(X) \arrow[d, "F(B)(\eta)"] \\
F(A)(Y) \arrow[rr, "F(t)(Y)"'] & & F(B)(Y)
\end{tikzcd}$$

for all t.

Therefore since $L^{Y,F}$ is the limit of $E_Y \circ F$, there exists a unique morphism τ such that the diagram

$$\begin{tikzcd}
L^{X,F} \arrow[d, "\tau"'] \arrow[dr, "F(A)(\eta)\circ\lambda^{X,F}(A)"] & \\
L^{Y,F} \arrow[r, "\lambda^{Y,F}(A)"'] & F(A)(Y)
\end{tikzcd}$$

commutes for all $A \in Ob(D)$ (The meaning of this diagram will be clear in the next step). Then let V map

$$V : \eta \mapsto \tau$$

It is then easily seen that V preserves identity. For preservation of composition, simply note the commutativity of

$$\begin{tikzcd}
L^{X,F} \arrow[r, "\lambda^{X,F}(A)"] \arrow[d, "V(\eta)"'] & F(A)(X) \arrow[d, "F(A)(\eta)"] \\
L^{Y,F} \arrow[r, "\lambda^{Y,F}(A)"] \arrow[d, "V(\mu)"'] & F(A)(Y) \arrow[d, "F(A)(\mu)"] \\
L^{Z,F} \arrow[r, "\lambda^{Z,F}(A)"'] & F(A)(Z)
\end{tikzcd}$$

Define λ_A which maps from $Ob(C)$ to $\mathrm{Hom}(\mathbb{A})$ by

$$\lambda_A : X \mapsto \lambda^{X,F}(A)$$

Then λ_A is a morphism from V to $F(A)$ in $Fct(C, \mathbb{A})$, due to the commutativity of

$$\begin{tikzcd}
V(X) \arrow[r, "\lambda^{X,F}(A)"] \arrow[d, "V(\eta)"'] & F(A)(X) \arrow[d, "F(A)(\eta)"] \\
V(Y) \arrow[r, "\lambda^{Y,F}(A)"'] & F(A)(Y)
\end{tikzcd}$$

Then we show that $(V, \{\lambda_A\}_A)$ is the limit of F.

For suppose that $t : A \to B$. We want the commutativity of the diagram in $Fct(C, \mathbb{A})$

$$\begin{tikzcd}
 & V \arrow[dl, Rightarrow, "\lambda_A"'] \arrow[dr, Rightarrow, "\lambda_B"] & \\
F(A) \arrow[rr, Rightarrow, "F(t)"'] & & F(B)
\end{tikzcd}$$

which, by substituting arbitrary X, we find that it commutes. So indeed, we have that the object is in the category $\mathbf{Lim}(F, D, Fct(C, \mathbb{A}))$.

Finally we need to show that it is terminal in this category. Suppose we have (U, u) in $\mathbf{Lim}(F, D, Fct(C, \mathbb{A}))$. This means that we have the commutative diagram

$$\begin{tikzcd}
 & U \arrow[dl, Rightarrow, "u_A"'] \arrow[dr, Rightarrow, "u_B"] & \\
F(A) \arrow[rr, Rightarrow, "F(t)"'] & & F(B)
\end{tikzcd}$$

Evaluating this at X, we get that $(U(X), \{u_A(X)\}_A)$ is in $\mathbf{Lim}(E_Y \circ F, D, \mathbb{A})$. This allows us to take unique $\gamma_X : U(X) \to L^{X,F}$ making the appropriate diagram commute. The map

$$\gamma : X \mapsto \gamma_X$$

obeys functoriality, because trivially, $V(\eta) \circ \gamma(X)$ and $\gamma(Y) \circ U(\eta)$ are both the unique morphism that, when composed with $\gamma_A(Y)$, gives $U(\eta) \circ u_B(Y) : U(X) \to F(A)(Y)$ for all $A \in Ob(C)$; indeed we see that $U(\eta) \circ u_B(Y)$ is iobviously n $\mathbf{Lim}(E_Y \circ F, D, \mathbb{A})$.

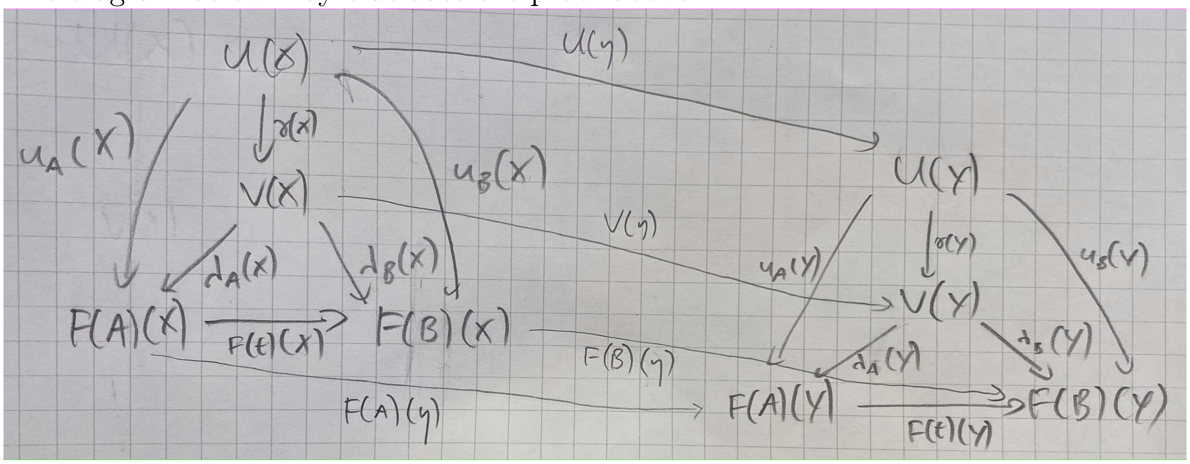
Then we obtain the commutativity of

$$\begin{tikzcd}
U \arrow[d, Rightarrow, "\gamma"'] \arrow[dr, Rightarrow, "u_A"] & \\
V \arrow[r, Rightarrow, "\lambda_A"'] & F(A)
\end{tikzcd}$$

due to the commutativity of the evaluation at X.

To show uniqueness of γ, evaluating at arbitrary X obtains diagrams in D, which, due to the universal property of $L^{X,F}$, obtains uniqueness. This completes the proof. The dual case concerning colimits is obtained by a dual proof. $\square$

The diagram below may elucidate the proof above



Because $\lim F$ is a functor from C to $\mathbb{A}$, one often expresses the above result by writing

$$(\lim_D F)(X) = L^{X,F} = \lim_D (E_X \circ F) = \lim_{A \in D} (F(A)(X))$$

and one says that "limits are computed pointwise". Note that to be precise, this does not give the complete picture, because we need to describe how the functor acts on morphisms as well.

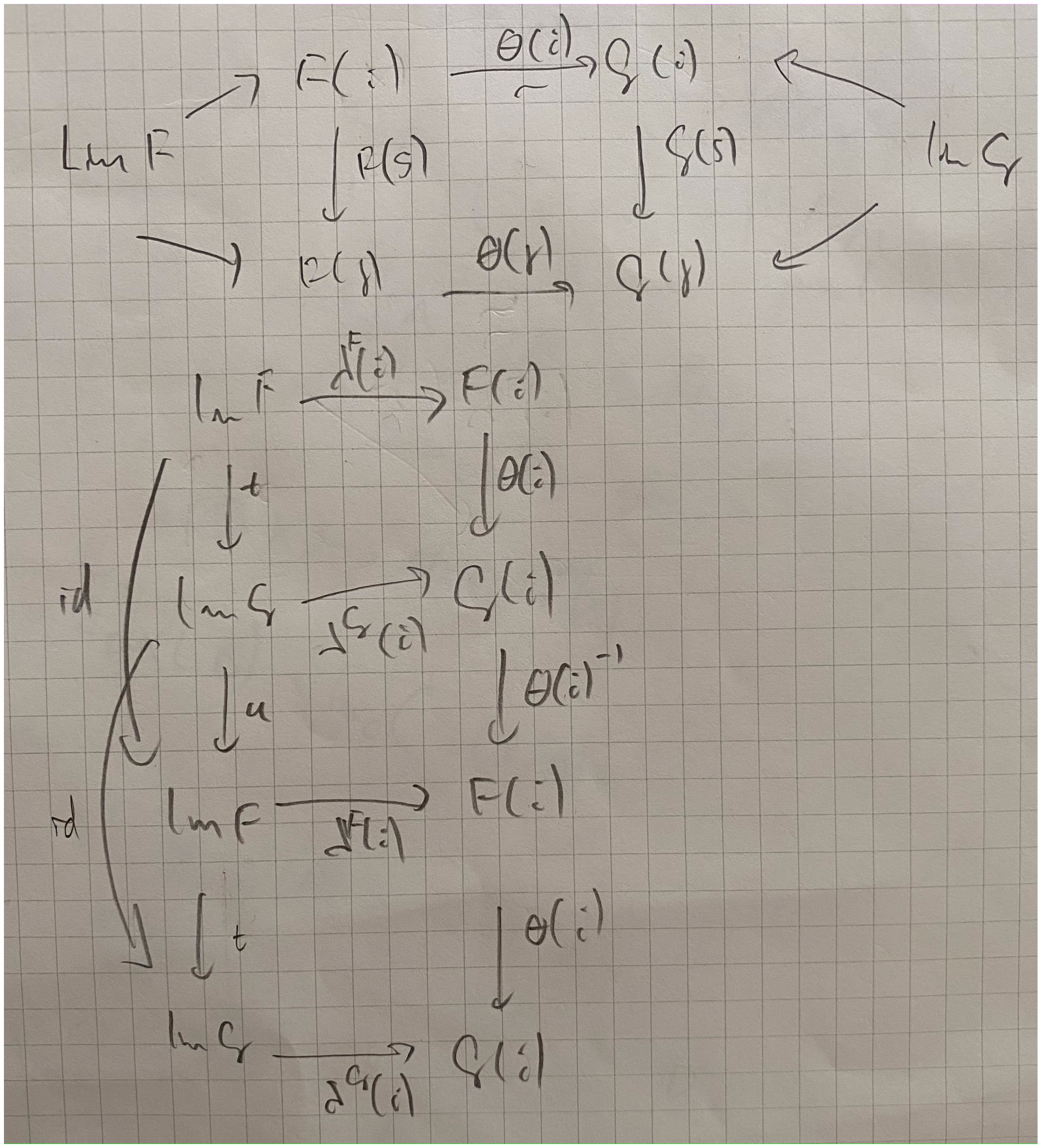
Corollary. *(Limits in product categories) If $\mathbb{A}$ is complete with respect to shapes C and D, then $Fct(C, \mathbb{A})$ is complete with respect to shape D. If $\mathbb{A}$ is cocomplete with respect to shapes C and D, then $Fct(C, \mathbb{A})$ is cocomplete with respect to shape D.*

Corollary. *If C is a small category, then $C^\vee = Fct(C, \mathbf{Set})$ and $C^\wedge = Fct(C^{op}, \mathbf{Set})$ admit limits and colimits of any small shape I.*

Proof. Because **Set** admits limits of small C and small I, by the above corollary, $Fct(C, \mathbf{Set})$ admits limits of shape I. The colimit case follows similarly. $\square$

Proposition 89. *Limits of isomorphic functors are isomorphic.*

Proof. Suppose we have $\theta : F \Longrightarrow G$ in the category $Fct(I, C)$. Observe the commutativity of



we see the unique existence of u and t satisfying the commutativity conditions, and hence obtain that $u \circ t = id$ and $t \circ u = id$. $\square$

6.7 Adjoints Preserve Limits

When $G : D \to C$ is a functor, and both C and D are complete with respect to I, then given any functor $\alpha : I \to D$, denote $(\lim \alpha, \{\lambda_i^\alpha\}_i)$ as its limit, and the functor $G \circ \alpha : I \to C$ has a limit, which we will denote $(\lim G\alpha, \{\lambda_i^{G\alpha}\}_i)$.

Given $G : D \to C$, we can consider two functors from $Fct(I, D)$ to C:

$$\alpha \mapsto G(\lim \alpha)$$

$$\theta \mapsto G(\lim \theta)$$

which is the composition of lim with G, and

$$\alpha \mapsto \lim(G\alpha)$$

$$\theta \mapsto \lim(G\theta)$$

where $G\theta$ is a map which takes $i \in Ob(I)$ to $G(\theta(i))$. Then by the definition of the limit functor, we obtain that the diagram

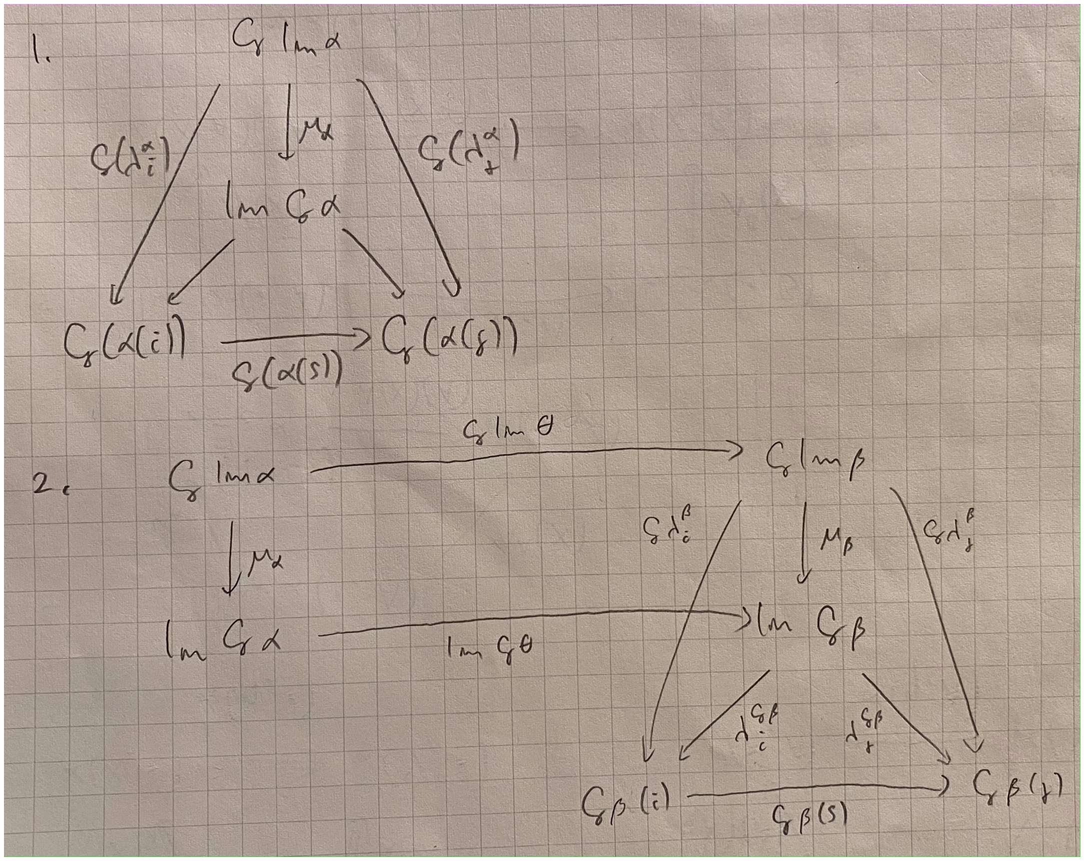
$$\begin{tikzcd}
\lim G\alpha \arrow[r, "\lim G\theta"] \arrow[d, "\lambda_i^{G\alpha}"'] & \lim G\beta \arrow[d, "\lambda_i^{G\beta}"] \\
G\alpha(i) \arrow[r, "G\theta(i)"'] & G\beta(i)
\end{tikzcd}$$

Now, if $\theta : \alpha \Longrightarrow \beta$ is a morphism of functors in $Fct(I, D)$, then we have

$$G(\lim \theta) : G(\lim \alpha) \to G(\lim \beta)$$

$$\lim(G\theta) : \lim(G\alpha) \to \lim(G\beta)$$

Then due to the commutativity, there exists μ_α making diagram 1 commute. Therefore diagram 2 commutes due to the universal property of $\lim G\beta$.



So we have a morphism of functors

$$G \lim \bullet \longrightarrow \lim G\bullet$$

in the category $Fct(I, D) \to C$. When this is an isomorphism, we shall say that "G preserves limits of shape I" or "G commutes with limits of shape I". In general, if G preserve limits of any shape, we shall simply state "G preserves limits".

In particular, if for any small category I, we have that G preserves limits of shape I, then we shall say that "G is continuous". That is, a continuous functor is one that preserves all small limits (this terminology is adopted from the topologists; recall that limits are preserved under continuous functions).

Theorem 90. *Right adjoints preserve limits.*

That is, if $G : D \to C$ is a right adjoint, and $\alpha : I \to D$ and $(L, \{\lambda_i\}_i)$ is the limit of α, then $(GL, \{G\lambda_i\}_i)$ is a limit of $G\alpha : I \to C$.

Notationally:

$$G \lim \alpha \approx \lim G\alpha$$

Remark. We emphasize here that the theorem states the existence of the limit of $G\alpha$ in the above proposition.

Proof. Suppose G is a right adjoint where $G : D \to C$. Take $F : C \to D$ such that $(F \dashv^{\gamma}_{\varepsilon} G)$. Suppose we have functor $\alpha : I \to D$. Then we shall show that if $(L, \{\lambda_i\}_i)$ is the limit of α, then $(GL, \{G\lambda_i\}_i)$ is a limit of $G\alpha : I \to C$. This hinges on how we bring diagrams into $\mathbf{Lim}(\alpha, I, D)$ using F.

Indeed we immediately see that $(GL, \{G\lambda_i\}_i)$ is an element of $\mathbf{Lim}(G\alpha, I, C)$. Suppose $(b, \{f_i\}_i)$ is an object in $\mathbf{Lim}(G\alpha, I, C)$.

Then $(Gb, \{\varepsilon_{\alpha(i)} \circ Gf_i\}_i)$ is an element of $\mathbf{Lim}(\alpha, I, D)$. For indeed, given any $s : j \to k$ we have the commutativity of

$$\begin{tikzcd}
Fb \arrow[r, "Ff_j"] \arrow[dr, "Ff_k"'] & FG\alpha(j) \arrow[r, "\varepsilon_{\alpha(j)}"] \arrow[d, "GF\alpha s"] & \alpha(j) \arrow[d, "\alpha(s)"] \\
& FG\alpha(k) \arrow[r, "\varepsilon_{\alpha(k)}"] & \alpha(k)
\end{tikzcd}$$

and therefore take $h : Fb \to L$ such that the diagram

$$\begin{tikzcd}
Fb \arrow[r, "h"] \arrow[d, "Ff_j"] & L \arrow[d, "\lambda_i"] \\
FG\alpha(j) \arrow[r, "\varepsilon_{\alpha(j)}"] & \alpha(j)
\end{tikzcd}$$

commutes. Therefore it follows that the diagram

$$\begin{tikzcd}
b \arrow[r, "\gamma_b"] \arrow[d, "f_j"'] & GFb \arrow[r, "Gh"] \arrow[d, "GFf_j"] & GL \arrow[d, "G\lambda_i"] \\
G\alpha(j) \arrow[r, "\gamma_{G\alpha(j)}"'] & GFG\alpha(j) \arrow[r, "G\varepsilon_{\alpha(j)}"'] & G\alpha(j)
\end{tikzcd}$$

commutes. Due to the unit-counit adjunction, we have that $G\varepsilon_{\alpha(j)} \circ \gamma_{G\alpha(j)} = id$, and so we obtain $f_j = G\lambda_i \circ (Gh \circ \gamma_b)$, which means that the cone factors through GL. We note here that $Gh \circ \gamma_b : b \to GL$ is the right adjunct of h.

To show uniqueness, suppose $\mu : b \to GL$ is a morphism from $(b, \{f_i\}_i)$ to $(GL, \{G\lambda_i\}_i)$. Then observing that we have isomorphism

$$\rho_{b,L} : \mathrm{Hom}_D(F(b), L) \approx \mathrm{Hom}_C(b, G(L))$$

we have that the left adjunct of μ is given by $\rho_{b,L}^{-1}(\mu) = \varepsilon_L \circ F\mu$. We want to show that $\varepsilon_L \circ F\mu = h$. So compose it with λ_i. By universality of h, it suffices to show that $\lambda_i \circ \varepsilon_L \circ F\mu = \varepsilon_{\alpha(j)} \circ Ff_j$. Now due to the commutativity of

$$\begin{tikzcd} FGL \arrow[r, "\varepsilon_L"] \arrow[d, "FG\lambda_j"] & L \arrow[d, "\lambda_j"] \\ FG\alpha(j) \arrow[r, "\varepsilon_{\alpha(j)}"] & \alpha(j) \end{tikzcd}$$

so we get $\lambda_j \circ \varepsilon_L \circ F\mu = \varepsilon_{\alpha(j)} \circ FG\lambda_j \circ F\mu = \varepsilon_{\alpha(j)} \circ F(G\lambda_j \circ \mu)$, and by the assumption that μ is a morphism from $(b, \{f_i\}_i)$ to $(GL, \{G\lambda_i\}_i)$, we have that $G\lambda_j \circ \mu = f_j$, so we obtain $\varepsilon_{\alpha(j)} \circ Ff_j$ which is what was desired. Hence $\varepsilon_L \circ F\mu = h$.

Applying $\rho_{b,L}$ to h gives its right adjunct, hence $\mu = \rho_{b,L}\rho_{b,L}^{-1}(\mu) = \rho_{b,L}(\varepsilon_L \circ F\mu) = \rho_{b,L}(h) = Gh \circ \gamma_b$ and the theorem is proved. $\square$

We supply another proof, in the case when I is small and C is locally small.

Proof. Suppose $G : D \to C$ is a right adjoint, and $\alpha : I \to D$. Suppose $X \in Ob(C)$. Then taking $F : C \to D$ as the left adjoint of G, we have by definiton, the isomorphism

$$\mathrm{Hom}_C(X, G(\lim \alpha)) \approx \mathrm{Hom}_D(FX, \lim \alpha)$$

functorial in X. Recall the functorial definition of the limit in the case where I is small and C is locally small: we have the isomorphism

$$\theta : \lim \mathrm{Hom}_D(A, \alpha\bullet) \to \mathrm{Hom}_D\,(A, \lim \alpha)$$

functorial in A. Therefore, we have

$$\mathrm{Hom}_D(FX, \lim \alpha) \approx \lim \mathrm{Hom}_D(FX, \alpha\bullet)$$

which, again, by using the definition of adjointness, we have

$$\lim \mathrm{Hom}_D(FX, \alpha\bullet) \approx \lim \mathrm{Hom}_C(X, G\alpha\bullet)$$

functorial in X, and finally due to the functorial definition of the limit,

$$\lim \mathrm{Hom}_C(X, G\alpha\bullet) \approx \mathrm{Hom}_C(X, \lim G\alpha)$$

functorial in X.

In conclusion, we have

$$\mathrm{Hom}_C(\bullet, G(\lim \alpha)) \approx \mathrm{Hom}_C(\bullet, \lim G\alpha)$$

which is precisely the evaluation of the Yoneda embedding at $G(\lim \alpha)$ and $\lim G\alpha$. As the embedding is conservative, we have

$$G(\lim \alpha) \approx \lim G\alpha$$

□

Corollary. *Left adjoints preserve colimits*

That is, if $F : C \to D$ is a left adjoint, and $\alpha : I \to C$ and $(L, \{\lambda_i\}_i)$ is the colimit of α, then $(FL, \{F\lambda_i\}_i)$ is a colimit of $F\alpha : I \to D$.

Proof. We recall that the colimit of $\alpha : I \to C$ is the limit of its cofunctor $\alpha^{\mathrm{op}} : I^{\mathrm{op}} \to C^{\mathrm{op}}$, which is the terminal object in $\mathbf{Lim}(\alpha^{\mathrm{op}}, I^{\mathrm{op}}, D^{\mathrm{op}})$. We have that $(F\alpha)^{\mathrm{op}} : I^{\mathrm{op}} \to D^{\mathrm{op}}$ and we want to show that $(FL, \{F\lambda_i\}_i)$ is a terminal object in $\mathbf{Lim}((F\alpha)^{\mathrm{op}}, I^{\mathrm{op}}, C^{\mathrm{op}})$. Note the following points

1. $F^{\mathrm{op}} \circ \alpha^{\mathrm{op}} = (F\alpha)^{\mathrm{op}}$
2. $(L, \{\lambda_i\}_i)$ is the limit of α^{op}, that is, it is terminal in $\mathbf{Lim}(\alpha^{\mathrm{op}}, I^{\mathrm{op}}, D^{\mathrm{op}})$
3. $F^{\mathrm{op}} : C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a right adjoint, and therefore it preserves limits
4. That is, $(F^{\mathrm{op}}L, \{F^{\mathrm{op}}\lambda_i\}_i)$ is terminal in $\mathbf{Lim}(F^{\mathrm{op}}\alpha^{\mathrm{op}}, I^{\mathrm{op}}, C^{\mathrm{op}}) = \mathbf{Lim}((F\alpha)^{\mathrm{op}}, I^{\mathrm{op}}, C^{\mathrm{op}})$
5. That is, $(F^{\mathrm{op}}L, \{F^{\mathrm{op}}\lambda_i\}_i) = (FL, \{F\lambda_i\}_i)$ is the colimit of $F\alpha$.

Suppose C and I are categories. Then denote the functor const : $C \to Fct(I, C)$ which associates

$$A \mapsto c_A$$

$$f \mapsto c_f$$

where c_A is the constant functor, and c_f is the map from $Ob(I)$ to $\mathrm{Hom}(C)$ that maps X to f. Then it is verified that this makes const a functor. It is also common in literature to denote $\Delta :=$ const and call this functor the "diagonal functor" or the "constant diagram functor of shape I". □

6.8 Yoneda Embedding Preserves and Reflects Small Limit Cones and Small Limit Cocones

In this section, we define, in precise terms, what we mean when we say that "F preserves limit cones" or "F preserves limit cocones" for functor F.

Denote I, C, and D as categories and $F : I \to C$ and $G : C \to D$ are functors. We do not necessarily require F to have a limit. Suppose $(a, \{m_i\}_i)$ is a cone of F. That is, we have the commutativity of

$$\begin{tikzcd} & a \arrow[dl, "m_j"'] \arrow[dr, "m_k"] & \\ F(j) \arrow[rr, "F(t)"'] & & F(k) \end{tikzcd}$$

in the category C. Applying G, which is a functor, we obtain the commutativity of

$$\begin{tikzcd} & G(a) \arrow[dl, "G(m_j)"'] \arrow[dr, "G(m_k)"] & \\ GF(j) \arrow[rr, "GF(t)"'] & & GF(k) \end{tikzcd}$$

We shall say that G preserves limit cones iff whenever F has a limit cone $(L, \{\lambda_i\}_i)$, the cone under the image of G, that is, $(GL, \{G\lambda_i\}_i)$, is a limit cone in the category $\mathbf{Lim}(G \circ F, I, D)$.

We shall say that G reflects limit cones iff whenever $(Ga, \{Gm_i\}_i)$ is a limit cone in the category $\mathbf{Lim}(G \circ F, I, D)$, the cone $(a, \{m_j\}_j)$ is a limit cone in the category $\mathbf{Lim}(F, I, C)$.

Proposition 91. *The Yoneda embedding preserves limit cones of small shapes. (It is continuous)*

Proof. Recall the functorial definition of limits. We say that $(L, \theta_L^{-1}(id_L))$ is the limit of $\alpha : I \to C$ iff we have

$$\theta : \lim_{C} \operatorname{Hom}_C(\bullet, \alpha\bullet) \to \operatorname{Hom}_C(\bullet, L)$$

is an isomorphism of functors in $Fct(C^{\mathrm{op}}, \mathbf{Set}) = C^{\wedge}$. ABC $\square$

Proposition 92. *The Yoneda embedding reflects limit cones of small shapes.*

abc

6.9 Global Definition of Limits

Proposition 93. *(Global Definition of Limits) If functor $F : Fct(I, C) \to C$ is the adjoint (right adjoint) of const, then it maps functor $\alpha : I \to C$ to its limit* $\lim \alpha$*. If F is the coadjoint (left adjoint) of const, then it maps functor $\alpha : I \to C$ to its colimit* colim α*.*

Proof. Suppose we have the isomorphism

$$\mathrm{Hom}_{Fct(I,C)}(\mathrm{const}\bullet, \bullet) \approx \mathrm{Hom}_C(\bullet, F\bullet)$$

in the category $Fct(C \times Fct(I, C), \mathbf{Set})$. Then we have

$$\rho_{X,\alpha} : \mathrm{Hom}_{Fct(I,C)}(\mathrm{const}_X, \alpha) \approx \mathrm{Hom}_C(X, F\alpha)$$

in $Fct(C, \mathbf{Set})$. In particular, we will show that putting $X = F\alpha$, we have that $(F\alpha, \rho^{-1}_{F\alpha}(id_{F\alpha}))$ is the desired terminal object in $\mathbf{Lim}(\alpha, I, C)$.

For indeed,

$$\rho^{-1}_{F\alpha,\alpha}(id_{F\alpha}) : \mathrm{const}_{F\alpha} \Longrightarrow \alpha$$

and thus $\mu := \rho^{-1}_{F\alpha,\alpha}(id_{F\alpha})$ is a map from $Ob(I)$ to $\mathrm{Hom}(C)$ such that given any $s : i \to j$ in I, the diagram

$$\begin{tikzcd} F\alpha \arrow[r, "\mu_i"] \arrow[d, "id_{F\alpha}"'] & \alpha(i) \arrow[d, "\alpha(s)"] \\ F\alpha \arrow[r, "\mu_j"] & \alpha(j) \end{tikzcd}$$

commutes, which shows that $(F\alpha, \mu)$ is indeed an object of $\mathbf{Lim}(\alpha, I, C)$. Further, suppose that (X, δ) is an object of $\mathbf{Lim}(\alpha, I, C)$. Then we immediately see that δ is a morphism from const_X to α and hence is in $\mathrm{Hom}_{Fct(I,C)}(\mathrm{const}_X, \alpha)$. We note the commutativity of

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{const}_{F\alpha}, \alpha) \arrow[r, "\rho_{F\alpha,\alpha}"] \arrow[d, "\circ c_{\rho_X(X,\delta)}"'] & \mathrm{Hom}(F\alpha, F\alpha) \arrow[d, "\circ \rho_X(X,\delta)"] \\ \mathrm{Hom}(\mathrm{const}_X, \alpha) \arrow[r, "\rho_{X,\alpha}"] & \mathrm{Hom}(X, F\alpha) \end{tikzcd}$$

$$\begin{tikzcd} \mu \arrow[r, mapsto, "\rho_{F\alpha,\alpha}"] \arrow[d, mapsto, "\circ c_{\rho_{X,\alpha}(\delta)}"'] & id_{F\alpha} \arrow[d, mapsto, "\circ \rho_{X,\alpha}(\delta)"] \\ \delta \arrow[r, mapsto, "\rho_{X,\alpha}"] & \rho_{X,\alpha}(\delta) \end{tikzcd}$$

in **Set**, of morphisms in C. Noting here that $\mu \circ c_{\rho_{X,\alpha}(\delta)}$ is a morphism of $Fct(I, C)$ which maps

$$j \mapsto \mu_j \circ \rho_{X,\alpha}(\delta)$$

the commutativity of the above diagrams begets the commutativity of the triangle

$$\begin{tikzcd} X \arrow[d, "\rho_{X,\alpha}(\delta)"'] \arrow[dr, "\delta_j"] & \\ F\alpha \arrow[r, "\mu_j"'] & \alpha(j) \end{tikzcd}$$

which shows the existence of a morphism from (X, δ) to $(F\alpha, \mu)$ in $\mathbf{Lim}(\alpha, I, C)$. We proceed to show that this morphism is unique.

Suppose that we have morphism $\tau : X \to F\alpha$ satisfying

$$\begin{tikzcd} X \arrow[d, "\tau"'] \arrow[dr, "\delta_j"] & \\ F\alpha \arrow[r, "\mu_j"'] & \alpha(j) \end{tikzcd}$$

Which is saying that $\delta = \mu \circ c_\tau$ which begets the commutativity of

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{const}_{F\alpha}, \alpha) \arrow[r, "\rho_{F\alpha,\alpha}"] \arrow[d, "\circ c_\tau"'] & \mathrm{Hom}(F\alpha, F\alpha) \arrow[d, "\circ\tau"] \\ \mathrm{Hom}(\mathrm{const}_X, \alpha) \arrow[r, "\rho_{X,\alpha}"] & \mathrm{Hom}(X, F\alpha) \end{tikzcd}$$

$$\begin{tikzcd} \mu \arrow[r, mapsto, "\rho_{F\alpha,\alpha}"] \arrow[d, mapsto, "\circ c_\tau"'] & id_{F\alpha} \arrow[d, mapsto, "\circ\tau"] \\ \delta \arrow[r, mapsto, "\rho_{X,\alpha}"] & \tau \end{tikzcd}$$

which means that $\rho_{X,\alpha}(\delta) = \tau$. This proves the first claim.

For the dual claim, we argue using the duality of the notions. Suppose F is the coadjoint of const. Note the following points.

1. We have that $F^{\mathrm{op}} : Fct(I, C)^{\mathrm{op}} \to C^{\mathrm{op}}$ is the adjoint of $\mathrm{const}^{\mathrm{op}}$
2. We recall the equality $Fct(I, C)^{\mathrm{op}} = Fct(I^{\mathrm{op}}, C^{\mathrm{op}})$
3. Then F maps functor $\alpha^{\mathrm{op}} : I^{\mathrm{op}} \to C^{\mathrm{op}}$, which is in $Fct(I^{\mathrm{op}}, C^{\mathrm{op}})$, to its limit $\lim(\alpha^{\mathrm{op}})$ which is, by definition, the colimit of α.

$\square$

Corollary. *If const has an adjoint, then C is complete with respect to diagram I. If const has a coadjoint, then C is cocomplete with respect to diagram I.*

Proposition 94. *(Global Definition of Limits; Converse Implication)*

Suppose C is locally small and I is small and C is complete with respect to shape I. Recall that we have functor $F : Fct(I, C) \to C$ which maps functors $\alpha : I \to C$ to one of its limits $\lim \alpha$ (use axiom of choice). When we have morphism of functors $\theta : \alpha \Longrightarrow \beta$, let us denote $(L_\alpha, \{\lambda_i^\alpha\}_i)$ as the limit of α. Then we have that $(L_\alpha, \{\theta_i \circ \lambda_i^\alpha\}_i)$ is in $\mathbf{Lim}(\beta, I, C)$ and therefore there exists a unique morphism from L_α to L_β in accordance with the limit property of L_β. Then let F associate θ to this morphism.

The limit functor F is then an adjoint to the functor const.

Proof. We would like to show the isomorphism

$$\mathrm{Hom}_{Fct(I,C)}(\mathrm{const}\bullet, \bullet) \approx \mathrm{Hom}_C(\bullet, F\bullet)$$

in the category $Fct(C \times Fct(I, C), \mathbf{Set})$. Suppose we have $X \in Ob(C)$ and $\alpha \in Ob(Fct(I, C))$. Then as previously discussed, we observe that if we have $\delta : \mathrm{const}_X \Longrightarrow \alpha$, we have the commutativity of

$$\begin{tikzcd} X \arrow[d, "\delta_k"'] \arrow[dr, "\delta_j"] & \\ \alpha(j) \arrow[r, "\alpha(s)"'] & \alpha(k) \end{tikzcd}$$

and hence we can associate it to a morphism from X to $F\alpha$, which we shall denote as $\rho_{X,\alpha}(\delta)$.

For surjectivity, suppose that we have morphism $t : X \to F\alpha$ in C. Denote $(L_\alpha, \{\lambda_i^\alpha\}_i)$ as the limit of α. Then we have that $\{\lambda_i^\alpha \circ t\}_i$ is in $\mathbf{Lim}(\alpha, I, C)$ and defines a morphism of functors $\gamma : \mathrm{const}(X) \Longrightarrow \alpha$ due to the commutativity of

$$\begin{tikzcd} X \arrow[r, "\lambda_j^\alpha \circ t"] \arrow[d, "id"'] & \alpha(j) \arrow[d, "\alpha(s)"] \\ X \arrow[r, "\lambda_k^\alpha \circ t"] & \alpha(k) \end{tikzcd}$$

By definition, then, γ maps to t.

To show injectivity of this map, suppose we have that $\rho_{X,\alpha}(\delta) = \rho_{X,\alpha}(\delta')$, where δ and δ' satisfy commutativity properties which make (X, δ) and (X, δ') elements of $\mathbf{Lim}(\alpha, I, C)$. Then this immediately obtains that $\lambda_i \circ \rho_{X,\alpha}(\delta) = \lambda_i \circ \rho_{X,\alpha}(\delta')$ for all $i \in Ob(I)$, which means that $\delta = \delta'$.

We now show functoriality in the arguments. Suppose we have $f : X \to Y$ in C and $\theta : \alpha \Longrightarrow \beta$ in $Fct(I, C)$. Then we want to show the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{const}_Y, \alpha) \arrow[r, "\rho_{Y,\alpha}"] \arrow[d, "\theta\circ\bullet\circ\mathrm{const}_f"'] & \mathrm{Hom}(Y, F\alpha) \arrow[d, "F\theta\circ\bullet\circ f"] \\
\mathrm{Hom}(\mathrm{const}_X, \beta) \arrow[r, "\rho_{X,\alpha}"] & \mathrm{Hom}(X, F\beta)
\end{tikzcd}$$

Suppose we have $\delta \in \mathrm{Hom}(\mathrm{const}_Y, \alpha)$. Going the lower path, we obtain $\theta \circ \delta \circ \mathrm{const}_f$, which maps from i to $\theta_i \circ \delta_i \circ f$. Observing that we have the commutativity of

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[r, "\delta_j"] & \alpha(j) \arrow[r, "\theta_j"] & \beta(j) \\
& Y \arrow[u, "id"] \arrow[r, "\rho_{Y,\alpha}(\delta)"'] & F\alpha \arrow[u, "\lambda_j^\alpha"] \arrow[r, "F\theta"'] & F\beta \arrow[u, "\lambda_j^\beta"]
\end{tikzcd}$$

for all j, we obtain the end result of $F\theta \circ \rho_{Y,\alpha}(\delta) \circ f$. We see that this is the same result as going the upper path. $\square$

6.10 Limits Commute with Limits, Colimits Commute with Colimits

Theorem 95. *Let C and D and arbitrary categories, and let $\mathbb{A}$ be a locally small category, such that it is complete with respect to C and to D. Then $\mathbb{A}$ is complete with respect to $C \times D$ and the limit of a functor taken in $C \times D$ coincides with the limit taken in C then D and the limit taken in D then C and . To be precise:*

Suppose $\alpha : C \times D \to \mathbb{A}$ is a functor. Recall (from the section on bifunctors) that we have an isomorphism of categories

$$\Xi_D : Fct(C \times D, \mathbb{A}) \to Fct(D, Fct(C, \mathbb{A}))$$

Then map α to its image under the isomorphism and denote it $\Xi_D\alpha$. Then denote its limit (which, as we recall, exists due to completeness) as

$(L_D\Xi_D\alpha, \lambda^{\Xi_D\alpha})$ which is a functor from C to $\mathbb{A}$, and the limit of $(L_D\Xi_D\alpha)$, which we denote as $(L_C L_D\Xi\alpha, \lambda^{L_D\Xi_D\alpha})$ is an object of $\mathbb{A}$.

Then we have isomorphism

$$L\alpha \approx L_C L_D \Xi_D \alpha$$

in $\mathbb{A}$.

Proof. We shall make use of the Yoneda Lemma in this proof. Denote $\{L\alpha, \lambda\}$ as the limit of α. Recall that the contravariant Yoneda Embedding is fully faithful. This implies that it is conservative.

We have

$$h^{\mathbb{A}} : \mathbb{A} \to Fct(\mathbb{A}^{\mathrm{op}}, \mathbf{Set})$$

and we want to show the existence of an isomorphism

$$\mathrm{Hom}_{\mathbb{A}}(\bullet, \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha) \approx \mathrm{Hom}_{\mathbb{A}}(\bullet, L\alpha)$$

in **Set** (Note that to use the Yoneda lemma, we require local smallness). That is, given any $\mathbb{Y}$ in $\mathbb{A}$, there is a bijection of sets

$$\mathrm{Hom}_{\mathbb{A}}(\mathbb{Y}, \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha) \approx \mathrm{Hom}_{\mathbb{A}}(\mathbb{Y}, L\alpha)$$

that is functorial in $\mathbb{Y}$. Explicitely, this bijection is obtained as follows. Given morphism $\mu : \mathbb{Y} \to \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha$, the indexed set $\{\lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu\}_X$ is a morphism of functors $\gamma : \Delta_D\mathbb{Y} \implies L\Xi_D\alpha$ in $Fct(C, \mathbb{A})$. Since $L\Xi_D\alpha$ is the limit of $\Xi\alpha$, this obtains an indexed set of morphisms of functors $\{\lambda_A^{\Xi_D\alpha} \circ \gamma\}_A$ which is a morphism of functors $\tau : \Delta_C(\Delta_D\mathbb{Y}) \implies \Xi_D\alpha$ in $Fct(D, Fct(C, \mathbb{A}))$, and such τ determines γ. Due to the isomorphism of categories $Fct(C \times D, \mathbb{A}) \approx Fct(D, Fct(C, \mathbb{A}))$, we have $\Xi_D^{-1}\tau : \Delta_{C\times D}\mathbb{Y} \implies \alpha$, which is an object in $\mathbf{Lim}(\alpha, C \times D, \mathbb{A})$; it is quickly verified that $\Xi\Delta_{C\times D}\mathbb{Y} = \Delta_C(\Delta_D\mathbb{Y})$. From this we are therefore able to uniquely associate $\rho : \mathbb{Y} \to L\alpha$, and ρ uniquely determines $\Xi_D^{-1}\tau$. So we have given a bijection. It remains to show functoriality.

Explicitely, this takes $\mu : \mathbb{Y} \to \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha$ to $\{\lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu\}_X$, which we denote as γ'. Then we get $\{\lambda_A^{\Xi_D\alpha} \circ \gamma\}_A$. This is a map such that for each $A \in Ob(D)$, we get a map from $Ob(C)$ to $\mathrm{Hom}(\mathbb{A})$ which maps X to $\lambda_A^{\Xi_D\alpha}(X) \circ \lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu$. This is τ, a morphism of $Fct(D, Fct(C, \mathbb{A}))$. Then $\Xi_D^{-1}\tau$ is a map that takes (X, A) to $\lambda_A^{\Xi_D\alpha}(X) \circ \lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu$.

Suppose we have $\nu : \mathbb{Y} \to \mathbb{Y}'$ in $\mathbb{A}^{\mathrm{op}}$; then the first functor turns this morphism to $\circ\nu$. This takes $\mu : \mathbb{Y} \to \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha$ to $\mu \circ \nu : \mathbb{Y}' \to \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha$.

Then this obtains $\{\lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu \circ \nu\}_X$, which we denote as γ'. Then we get $\{\lambda_A^{\Xi_D\alpha} \circ \gamma'\}_A$. For each $A \in Ob(D)$, we get a map from $Ob(C)$ to $\mathrm{Hom}(\mathbb{A})$ which maps X to $\lambda_A^{\Xi_D\alpha}(X) \circ \lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu \circ \nu$. Denote this as τ'. Applying Ξ_D^{-1}, we obtain $\Xi_D^{-1}\tau' : \Delta_{C\times D}\mathbb{Y} \Longrightarrow \alpha$, which obtains ρ'. We would like to show $\rho' = \rho \circ \nu$. This can be done by using explicitely the definition of Ξ_D.

We see that $\Xi_D^{-1}\tau'$ is a map that takes (X, A) to $\lambda_A^{\Xi_D\alpha}(X) \circ \lambda_X^{\mathrm{L}_D\Xi_D\alpha} \circ \mu \circ \nu$. Then since ρ' is the **unique** morphism which makes the diagram

$$\begin{tikzcd}
\mathbb{Y}' \arrow[d, "\rho'"'] \arrow[dr, "\lambda_A^{\Xi_D\alpha}(X)\circ\lambda_X^{\mathrm{L}_D\Xi_D\alpha}\circ\mu\circ\nu"] & \\
\alpha(X, A) \arrow[r, "\alpha(s,r)"'] & \alpha(Y, B)
\end{tikzcd}$$

commutative, and ρ makes the diagram

$$\begin{tikzcd}
\mathbb{Y}' \arrow[d, "\nu"] & \\
\mathbb{Y} \arrow[d, "\rho"'] \arrow[dr, "\lambda_A^{\Xi_D\alpha}(X)\circ\lambda_X^{\mathrm{L}_D\Xi_D\alpha}\circ\mu\circ\nu"] & \\
\alpha(X, A) \arrow[r, "\alpha(s,r)"'] & \alpha(Y, B)
\end{tikzcd}$$

we have no other choice than to conclude $\rho' = \rho\circ\nu$. This proves functoriality, which means that the isomorphism

$$\mathrm{Hom}_{\mathbb{A}}(\bullet, \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha) \approx \mathrm{Hom}_{\mathbb{A}}(\bullet, L\alpha)$$

is proven. Since h^C reflects isomorphisms, we obtain

$$L\alpha \approx \mathrm{L}_C\mathrm{L}_D\Xi_D\alpha$$

Noting that $C \times D$ is more formally defined as the product of two indexed categories (in contrast to the ordered pair definition of the product), $M_1 = C$, $M_2 = D$, and in our proof, had simply taken some $i = 1, 2$ for the isomorphism

$$\Xi : Fct(M_1 \times M_2, \mathbb{A}) \to Fct(M_j, Fct(M_i, \mathbb{A}))$$

where $j \neq i$. The choice of i was arbitrary, so we obtain the same isomorphism

$$L\alpha \approx \mathrm{L}_D\mathrm{L}_C\Xi_C\alpha$$

where we take the limit of α in C then in D. $\square$

Remark. It needs to be noted here that the proof above does not say anything about the existence of the limit of α, so what was proved was weaker than the original theorem.

The following is the second proof that we give by explicitely showing that $\mathrm{L}_C\mathrm{L}_D\Xi_D\alpha$ is a limit of α.

Proof. Suppose C, D, $\mathbb{A}$ are categories and $\alpha : C \times D \to \mathbb{A}$ is a functor. Recall that we have an isomorphism of categories

$$\Xi : Fct(C \times D, \mathbb{A}) \to Fct(D, Fct(C, \mathbb{A}))$$

Then map α to its image under the isomorphism and denote it $\Xi\alpha$. Then denote its limit as $(\mathrm{L}\Xi\alpha, \lambda)$ is an functor from C to $\mathbb{A}$, and the limit of $(\mathrm{L}\Xi\alpha)$, which we denote as $(\mathrm{LL}\Xi\alpha, \Lambda)$ is an object of $\mathbb{A}$. Then we are able to associate this pair of objects with the limit of α in the following sense. (I include drawn diagrams because I am not bothered to typset them)

1.

$$\begin{tikzcd}
 & LL\Xi\alpha \arrow[dl, "\Lambda_X"'] \arrow[dr, "\Lambda_Y"] & \\
(L\Xi\alpha)(X) \arrow[rr, "(L\Xi\beta)(\eta)"'] & & (L\Xi\alpha)(Y)
\end{tikzcd}$$

2.

$$\begin{tikzcd}
\Xi\alpha(A)(X) \arrow[r, "\Xi\alpha(t)(X)"] \arrow[d, "\Xi\alpha(A)(\eta)"] & \Xi\alpha(B)(X) \arrow[d, "\Xi\alpha(B)(\eta)"] \\
\Xi\alpha(A)(Y) \arrow[r, "\Xi\alpha(t)(Y)"] & \Xi\alpha(B)(Y)
\end{tikzcd}$$

3.

$$\begin{tikzcd}
 & L\Xi\alpha \arrow[dl, "\lambda_A"'] \arrow[dr, "\lambda_B"] & \\
\Xi\alpha(A) \arrow[rr, "\Xi\alpha(t)"'] & & \Xi\alpha(B)
\end{tikzcd}$$

4.

$$\begin{tikzcd}
 & L\Xi\alpha(X) \arrow[r, "\lambda_A(X)"] \arrow[d, "L\Xi\alpha(\eta)"] & \Xi\alpha(A)(X) \arrow[r, "\Xi\alpha(t)(X)"] \arrow[d, "\Xi\alpha(A)(\eta)"] & \Xi\alpha(B)(X) \arrow[d, "\Xi\alpha(B)(\eta)"] \\
LL\Xi\alpha \arrow[ur, "\Lambda_X"] \arrow[r, "\Lambda_Y"'] & L\Xi\alpha(Y) \arrow[r, "\lambda_A(Y)"'] \arrow[rr, bend right, "\lambda_B(Y)"'] & \Xi\alpha(A)(Y) \arrow[r, "\Xi\alpha(t)(Y)"'] & \Xi\alpha(B)(Y)
\end{tikzcd}$$

5.

$$\begin{tikzcd}
 & LL\Xi\alpha \arrow[dl, "\Lambda_X"'] \arrow[dr, "\Lambda_Y"] & \\
L\Xi\alpha(X) \arrow[d, "\lambda_A(X)"'] & & L\Xi\alpha(Y) \arrow[d, "\lambda_B(Y)"] \\
\Xi\alpha(A)(X) \arrow[rr] & & \Xi\alpha(B)(Y)
\end{tikzcd}$$

$$\Xi\alpha(t)(Y) \circ \Xi\alpha(A)(\eta)$$
$$= \alpha(B)(\eta) \circ \Xi\alpha(t)(X)$$
$$= \alpha(\eta, t)$$

We observe the commutativity of diagram 1 given any $\eta : X \to Y$ in C. We have that $\Xi\alpha(t) : \Xi\alpha(A) \to \Xi\alpha(B)$ is a morphism of functors in $Fct(C, A)$, given any morphism $t : A \to B$ in D, that is, we have the commutativity of diagram 2. Noting that we have $(\mathrm{L}\Xi\alpha, \lambda) \in Fct(C, \mathbb{A})$, we have the commutativity of diagram 3 in the category $Fct(C, \mathbb{A})$.

We have that $\Lambda_X : \mathrm{LL}\Xi\alpha \to \mathrm{L}\Xi\alpha(X)$ and because λ_A is a morphism of functors, we have $\lambda_A(X) : \mathrm{L}\Xi\alpha(X) \to \Xi\alpha(A)(X) = \alpha(X, A)$.

Denote the $\pi_{X,A} := \lambda_A(X) \circ \Lambda_X$. We shall show that $(\mathrm{LL}\Xi\alpha, \pi)$ is the limit of α. From diagrams 1, 2, and 3, we obtain the commutativity of diagram 4, from which it follows that diagram 5 is commutative.

We now show universality of the given object. Suppose $(\mathbb{Y}, \{f_{X,A}\}_{X,A})$ is in $\mathbf{Lim}(\alpha, C \times D, \mathbb{A})$. We desire to show the existence and uniqueness of morphism $b : \mathbb{Y} \to \mathrm{LL}\Xi\alpha$ in diagram 1.

1.

$$\begin{tikzcd}
\mathbb{Y} \arrow[d, dashed, "\exists b"'] \arrow[dr, squiggly] \arrow[drr, "f_{X,A}"] & & \\
LL\Xi\alpha \arrow[r, "\Lambda_X"'] & L\Xi\alpha(X) \arrow[r, "\lambda_A(X)"'] & \Xi\alpha(A)(X) = \alpha(X,A)
\end{tikzcd}$$

2.

$$\begin{tikzcd}
\mathbb{Y} \arrow[d] \arrow[dr, "f_{X,A}"] & \\
L\Xi\alpha(X) \arrow[r, "\lambda_A(X)"'] & \alpha(X,A)
\end{tikzcd}$$

3.

$$\delta^A : \Delta\mathbb{Y} \Longrightarrow \Xi\alpha(A)$$
$$X \longmapsto f_{X,A}$$

4.

$$\begin{tikzcd}
\mathbb{Y} \arrow[r, "\delta_X = f_{X,A}"] \arrow[d, "id"] & \Xi\alpha(A)(X) \arrow[d, "\Xi\alpha(A)(\eta) = \alpha(\eta, A)"] \\
\mathbb{Y} \arrow[r, "\delta_Y = f_{Y,A}"'] & \Xi\alpha(A)(Y)
\end{tikzcd}$$

5.

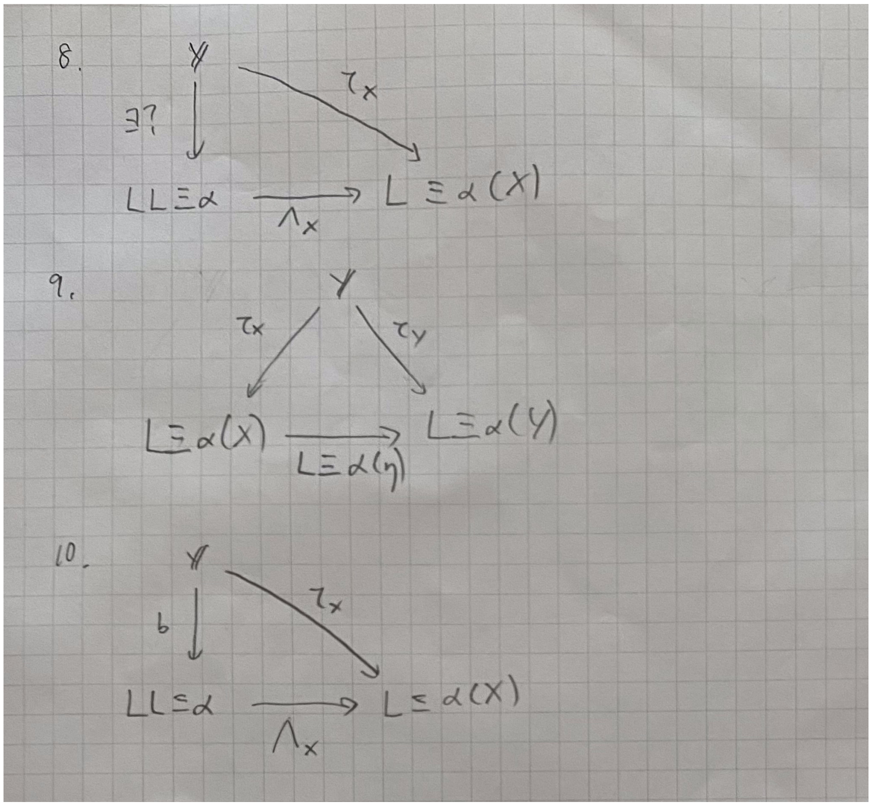
$$\begin{tikzcd}
& \Delta\mathbb{Y} \arrow[dl, Rightarrow, "\delta^A"'] \arrow[dr, Rightarrow, "\delta^B"] & \\
\Xi\alpha(A) \arrow[rr, Rightarrow, "\Xi\alpha(t)"'] & & \Xi\alpha(B)
\end{tikzcd}$$

6.

$$\begin{tikzcd}
& \mathbb{Y} \arrow[dl, "f_{X,A}"'] \arrow[dr, "f_{X,B}"] & \\
\alpha(X,A) \arrow[rr, "\alpha(X,t)"'] & & \alpha(X,B)
\end{tikzcd}$$

7.

$$\begin{tikzcd}
\Delta\mathbb{Y} \arrow[d, Rightarrow, "\tau"'] \arrow[dr, Rightarrow, "\delta^A"] & \\
L\Xi\alpha \arrow[r, Rightarrow, "\lambda_A"'] & \Xi\alpha(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Y} \arrow[d, "\tau_X"'] \arrow[dr, "f_{X,A}"] & \\
L\Xi\alpha(X) \arrow[r, "\lambda_A(X)"'] & \Xi\alpha(A)(X)
\end{tikzcd}$$



We shall first focus on constructing the commutative diagram in 2. We observe given any $A \in Ob(D)$, that δ^A as defined in 3 is in fact a morphism of functors due to the commutativity of 4. Then given objects $A, B \in Ob(D)$, we see that the diagram in 5 commutes in $Fct(C, \mathbb{A})$ as the evaluation of the morphisms at X gives the commutative diagram in 6. This begets the commutative diagram in 7. Noticing that when we evaluate τ at X, we naturally want a commutative diagram as in 8. Reminding ourselves that τ is a morphism of functors from $\Delta\mathbb{Y}$ to $\mathrm{L}\Xi\alpha$, we obtain the commutativity of 9, from which we deduce the existence of b in 10. This in turn, shows that 1 commutes.

For uniqueness, suppose that b' satisfies the same diagram as b in 1. Noting that the construction of δ^A and τ is independent of b and b', we have that $\{\Lambda_X \circ b\}_X$ satisfies the same commutativity relation as τ does in 7 and is therefore a morphism of functors $\Delta\mathbb{Y} \Longrightarrow \mathrm{L}\Xi\alpha$. The universal property of $\mathrm{L}\Xi\alpha$ therefore obtains $\tau = \{\Lambda_X \circ b\}_X$ and due to 10, we obtain $b = b'$ due to the universal property of $\mathrm{LL}\Xi\alpha$.

So as we have claimed, the pair $(\mathrm{LL}\Xi\alpha, \pi)$ is the limit of α. That is, the limit of α exists. We can then write

$$\mathrm{L}\alpha \approx \mathrm{L}_C \mathrm{L}_D \Xi_D \alpha$$

since C and D could be swapped, we also have

$$\mathrm{L}_C \mathrm{L}_D \Xi_D \alpha \approx \mathrm{L}_D \mathrm{L}_C \Xi_C \alpha$$

□

We give a third proof using the fact that right adjoints preserve limits.

Proof. Suppose $\alpha : C \times D \to \mathbb{A}$ is a functor. We note the following points.

1. We recall that $\mathbb{A}$ is complete with respect to the shape $Fct(C, \mathbb{A})$.

2. We therefore have that the limit functor $\lim_C : Fct(C, \mathbb{A}) \to \mathbb{A}$ exists. It is a right adjoint.

3. $\Xi_D \alpha : D \to Fct(C, \mathbb{A})$ is a functor, with limit $\lim_D \Xi_D \alpha$, an element of $Fct(C, \mathbb{A})$. Taking its limit, we get $\lim_C \lim_D \Xi_D \alpha = \mathrm{L}_C \mathrm{L}_D \Xi_D \alpha$.

4. Adjoints preserve limits. That is, we get $\lim_C(\lim_D \Xi_D \alpha) \approx \lim_D(\lim_C \circ \Xi_D \alpha)$

5. We want to show that $\lim_C \circ \Xi_D \alpha \approx \lim_C \Xi_C \alpha$ in $Fct(D, \mathbb{A})$. Substitute $A \in Ob(D)$ into both sides to obtain

$$\lim_C (\alpha(\bullet, A)) \approx (\lim_C \Xi \alpha)(A)$$

   Due to the definition of the limit, we have that diagram 1 and 2 commute for all $\eta : X \to Y$. Substituting A in diagram 2 we get diagram 3. Then due to the universal property of the limit, we are able to take unique

$$\mu_A : \lim_C (\alpha(\bullet, A)) \to (\lim_C \Xi \alpha)(A)$$

   such that diagram 4 commutes for all $X \in Ob(C)$; we see that μ_A is an isomorphism. To show that this is functorial in A, suppose $t : A \to B$. Recall that the definition of the limit functor, that the left hand side obtains

$$\lim_C (\alpha(\bullet, t)) : X \mapsto \alpha(X, t) \circ \Lambda^A(X)$$

and the right hand side obtains $(\lim_C \Xi\alpha)(t)$ which is the map which makes the diagram in 5 commute. Then observe, in diagram 6, that

$$\lambda(X)(B) \circ \mu_B \circ \lim_C \alpha(\bullet, t) = \lambda(Y)(B) \circ \mu_B \circ \lim_C \alpha(\bullet, t)$$

$$\lambda(X)(B) \circ \lim_C \Xi\alpha(t) \circ \mu_A = \lambda(Y)(B) \circ \lim_C \Xi\alpha(t) \circ \mu_A$$

and therefore by the universal property of $\lim_C \Xi\alpha(B)$, we obtain that the whole of diagram 6 commutes, which proves functoriality, as desired.

6. Since isomorphic functors have isomorphic limits, the isomorphism

$$\lim_D(\lim_C \circ \Xi_D\alpha) \approx \lim_D(\lim_C \Xi_C\alpha)$$

from which we obtain

$$\lim_C(\lim_D \Xi_D\alpha) \approx \lim_D(\lim_C \Xi_C\alpha)$$

1.

$$\begin{tikzcd}
 & \lim_C \alpha(\cdot, A) \arrow[dl, "\Lambda^A(X)"'] \arrow[dr, "\Lambda^A(Y)"] & \\
\alpha(X, A) \arrow[rr, "\alpha(\eta, A)"'] & & \alpha(Y, A)
\end{tikzcd}$$

2.

$$\begin{tikzcd}
 & \lim_C \Xi_C \alpha \arrow[dl, Rightarrow, "\lambda(X)"'] \arrow[dr, Rightarrow, "\lambda(Y)"] & \\
\alpha(X, \cdot) \arrow[rr, Rightarrow, "\alpha(\eta, \cdot)"'] & & \alpha(Y, \cdot)
\end{tikzcd}$$

3.

$$\begin{tikzcd}
 & \lim_C \Xi_C \alpha(A) \arrow[dl, "\lambda(X)(A)"'] \arrow[dr, "\lambda(Y)(A)"] & \\
\alpha(X, A) \arrow[rr, "\alpha(\eta, A)"'] & & \alpha(Y, A)
\end{tikzcd}$$

4.

$$\begin{tikzcd}
 & \lim_C \alpha(\cdot, A) \arrow[dl, "\mu_A"'] \arrow[dr, "\Lambda^A(X)"] & \\
\lim_C \Xi_C \alpha \arrow[rr, "\lambda(X)(A)"'] & & \alpha(X, A)
\end{tikzcd}$$

5.

$$\begin{tikzcd}
\lim_C \alpha(\cdot, A) \arrow[r, "\lim_C \alpha(\cdot, t)"] \arrow[d, "\Lambda^A(X)"] & \lim_C \alpha(\cdot, B) \arrow[d, "\Lambda^B(X)"] \\
\alpha(X, A) \arrow[r, "\alpha(X, t)"'] & \alpha(X, B)
\end{tikzcd}$$

6.

$$\begin{tikzcd}
 & \lim_C \alpha(\cdot, A) \arrow[d, "\mu_A"] \arrow[ddl, bend right, "\Lambda^A(X)"'] \arrow[ddr, bend left, "\Lambda^A(Y)"] \arrow[rrr, "\lim_C \alpha(\cdot, t)"] & & & \lim_C \alpha(\cdot, B) \arrow[d, "\mu_B"] \arrow[ddl, bend right, "\Lambda^B(X)"'] \arrow[ddr, bend left, "\Lambda^B(Y)"] & \\
 & \lim_C \Xi \alpha(A) \arrow[dl, "\lambda(X)(A)"'] \arrow[dr, "\lambda(Y)(A)"] \arrow[rrr, "\lim_C \Xi \alpha(t)"] & & & \lim_C \Xi \alpha(B) \arrow[dl, "\lambda(X)(B)"'] \arrow[dr, "\lambda(Y, B)"] & \\
\alpha(X, A) \arrow[rr, "\alpha(\eta, A)"'] \arrow[rrr, bend right, "\alpha(X, t)"'] & & \alpha(Y, A) \arrow[rrr, bend left, "\alpha(Y, t)"] & \alpha(X, B) \arrow[rr, "\alpha(\eta, B)"'] & & \alpha(Y, B)
\end{tikzcd}$$

To compare the limit with $\lim \alpha$, ABCABC $\square$

6.11 A Category that has Products and Equalizers also has Limits

It is noted that a category that has limits also has products and equalizers. This is because a product is a limit, and the equalizer category $\mathbf{Eq}_C(f, g)$ is isomorphic to the limit category of some functor $\mathbf{Lim}(\alpha, I, C)$. Since the terminal object exists in $\mathbf{Lim}(\alpha, I, C)$, it exists in $\mathbf{Eq}_C(f, g)$ and therefore the equalizer of any two morphisms exists. So a category with limits has products and equalizers. The rest of this section shall be devoted to proving the converse of this statement.

We shall give two proofs for the result in the title of this section. It will also follow that a cateogory that has coproducts and coequalizers also has colimits.

Recall that the category of sets has products and equalizers. Given a category I, denote τ as the map which takes a morphism $s : j \to k$ in I to the object k in I, and σ as the map which takes $s : j \to k$ in I to the object j in I.

Theorem 96. *Let C denote any category that has products for any collection of objects, and equalizers for any pair of morphisms. For small category I, and functor $\alpha : I \to C$, denote*

$$\prod_i \alpha i$$

$$\{\pi_i\}_i$$

as the product of the indexed set $\{\alpha i\}_{i \in Ob(I)}$. Denote

$$\prod_{\substack{s:i\to j \\ s\in Hom(I)}} \alpha j$$

$$\{\pi_{s:i\to j}\}_s$$

as the product of the indexed set $\{\alpha\tau s\}_{s\in Hom(I)}$.

If we define t and u such that

$$t, u : \prod_i \alpha i \longrightarrow \prod_{s:i\to j} \alpha j$$

where t and u are the unique morphisms such that for all $s : i \to j$, the diagrams

$$\begin{tikzcd} \prod_{i'} \alpha i' \arrow[d, "t"'] \arrow[dr, "\pi_j"] & \\ \prod_{s':i' \to j'} \alpha j' \arrow[r, "\pi_s"'] & \alpha j \end{tikzcd}$$

$$\begin{tikzcd} \prod_{i'} \alpha i' \arrow[r, "\pi_i"] \arrow[d, "u"'] & \alpha i \arrow[d, "\alpha s"] \\ \prod_{s':i' \to j'} \alpha j' \arrow[r, "\pi_s"'] & \alpha j \end{tikzcd}$$

commute; that is, for all s:

$$\pi_{(s:i \to j)} \circ t = \pi_j$$

$$\pi_{(s:i \to j)} \circ u = \alpha s \circ \pi_i$$

Denoting (E, e) as the equalizer of (t, u), we have that $(E, \{\pi_i \circ e\}_i)$ is a cone and is in fact the terminal cone.

Proof. That it is in fact a cone easily follows from observing the commutativity of the diagrams given above. To show its universality, suppose $(X, \{m_i\}_i)$ is a cone of α. Then factor this through $\prod_{i'} \alpha i'$ to obtain $\{m_i\}_i = \{\pi_i \circ q\}_i$. Then due to the commutativity properties of cones, the definitions of t and u obtain the equality

$$\pi_s \circ t \circ q = \pi_s \circ u \circ q$$

for all $s : i \to j$. By the definition of the product, this means that

$$t \circ q = u \circ q$$

which means that q is in $\mathbf{Eq_{Set}}(t, u)$. Therefore, we are allowed to take $\mu : X \to E$ such that the diagram

$$\begin{tikzcd} X \arrow[d, "\mu"'] \arrow[dr, "q"] & \\ E \arrow[r, "e"'] & \prod_{i'} \alpha i' \end{tikzcd}$$

commutes, and we immediately see that μ is a morphism from the cone $(X, \{m_i\}_i)$ to the cone $(E, \{\pi_i \circ e\}_i)$. It now only remains to show the uniqueness of μ. Indeed, if η is a morphism from the cone $(X, \{m_i\}_i)$ to the cone $(E, \{\pi_i \circ e\}_i)$, then we have

$$\pi_i \circ e \circ \eta = m_i = \pi_i \circ q$$

which, due to the universal property of the product, immediately obtains

$$e \circ \eta = q = e \circ \mu$$

which, due to the universal property of the equalizer, immedaitely obtains

$$\eta = \mu$$

□

For the second proof, we shall derive the result from the simpler case where $C = \mathbf{Set}$, and using the Yoneda lemma to generalize the result.

Proof. First we show that the above theorem holds for $C = \mathbf{Set}$. Recall that the limit in **Sets** for functor α is given by

$$\lim \alpha := \left\{ \{x_i\}_i \in \prod_i \alpha(i) \mid \alpha(s)\,(x_j) = x_k \text{ for all } s \in \mathrm{Hom}_I(j,k) \right\}$$

Now t and u as defined by the above theorem is given as

$$t : \{x_{i'}\}_{i'} \mapsto \{x_{j'}\}_{s':i' \to j'}$$

$$u : \{x_{i'}\}_{i'} \mapsto \{\alpha(s')(x_{j'})\}_{s':i' \to j'}$$

which implies

$$\mathrm{Eq}(t,u) = \left\{ \{x_i\}_i \in \prod_i \alpha(i) \mid \{x_{j'}\}_{s':i' \to j'} = \{\alpha(s')(x_{i'})\}_{s':i' \to j'} \right\}$$

which is immediately seen to be equal to $\lim \alpha$.

We shall reduce the problem to the category of sets using the Yoneda lemma. Denote y as the contravariant Yoneda embedding. Recall that **Set** is admits limits and colimits with respect to any small category. Recall that this in turn implies that $Fct(C, \mathbf{Set})$ admits limits and colimits with respect to any small category, and limits are computed pointwise.

For small category I, and functor $\alpha : I \to C$, denote

$$\prod_i \alpha i$$

$$\{\pi_i\}_i$$

as the product of the indexed set $\{\alpha i\}_{i \in Ob(I)}$. Denote

$$\prod_{\substack{s:i\to j\\ s\in \mathrm{Hom}(I)}} \alpha j$$

$$\{\pi_{s:i\to j}\}_s$$

as the product of the indexed set $\{\alpha \tau s\}_{s\in \mathrm{Hom}(I)}$. Both of these products are objects of the category C.

If we define t and u such that

$$t, u : \prod_i \alpha i \longrightarrow \prod_{s:i\to j} \alpha j$$

$$\pi_{(s:i\to j)} \circ t = \pi_j$$

$$\pi_{(s:i\to j)} \circ u = \alpha s \circ \pi_i$$

in C, for all $s : i \to j$ in I. Then the equalizer of t and u exists; denote it as (E, e).

The Yoneda embedding preserves limits of small shapes. We recall that equalizers and products are small limits. Therefore $(y(E), y(e))$ is the equalizer of $y(t)$ and $y(u)$ in the category $Fct(C^{\mathrm{op}}, \mathbf{Set})$.

Since the Yoneda embedding preserves cones and cocones of small shapes, we have that

$$y \prod_i \alpha i = \prod_i y\alpha i$$

$$\{y\pi_i\}_i$$

$$y \prod_{\substack{s:i\to j\\ s\in \mathrm{Hom}(I)}} \alpha j = \prod_{\substack{s:i\to j\\ s\in \mathrm{Hom}(I)}} y\alpha j$$

$$\{y\pi_{s:i\to j}\}_s$$

are the products of $\{y\alpha i\}_i$ and $\{y\alpha i\}_{s:i\to j}$, respectively, living in the category $Fct(C^{\mathrm{op}}, \mathbf{Set})$, where we have $y \circ \alpha : I \to Fct(C^{\mathrm{op}}, \mathbf{Set})$. The limit of this functor exists and is computed pointwise; that is, for $X \in Ob(C)$:

$$(\lim y\alpha)(X) := \lim_{i\in I}(y\alpha(i)(X)) = \lim_I E_X \circ y\alpha$$

in **Set**. Now again, since limits are computed pointwise, we have that evaluating the products at $X \in Ob(C)$:

$$\left(\prod_i y\alpha i\right)(X) = \prod_i (y\alpha(i)(X))$$

$$\{y\pi_i(X)\}_i$$

$$\left(\prod_{\substack{s:i\to j\\ s\in\mathrm{Hom}(I)}} y\alpha j\right) = \prod_{\substack{s:i\to j\\ s\in\mathrm{Hom}(I)}} y\alpha(j)(X)$$

$$\{y\pi_{s:i\to j}(X)\}_s$$

are the products of $\{(E_X \circ y\alpha)(i)\}_i$ and $\{(E_X \circ y\alpha)(j)\}_{s:i\to j}$, respectively, in **Set**.

Now obviously,

$$y\pi_{(s:i\to j)} \circ yt = y\pi_j$$

$$y\pi_{(s:i\to j)} \circ yu = y\alpha s \circ y\pi_i$$

in $Fct(C^{\mathrm{op}}, \mathbf{Set})$. Noting that these are morphisms of functors, evaluating this at $X \in Ob(C)$ obtains

$$y\pi_{(s:i\to j)}(X) \circ yt(X) = y\pi_j(X)$$

$$y\pi_{(s:i\to j)}(X) \circ yu(X) = y\alpha s(X) \circ y\pi_i(X)$$

in **Set** for each $s : i \to j$.

Since we are dealing with the category of sets, it follows from the previous lemma that the set theoretic equalizer of $yt(X)$ and $yu(X)$, which we shall denote as (Q_X, q_X), is the limit of $E_X \circ y\alpha : I \to \mathbf{Set}$. Again, limits are computed pointwise, so we have that the limit of $y\alpha$ exists and is given by

$$(\lim_I y\alpha)(X) = Q_X$$

Since the Yoneda embedding reflects limits, we have that $\lim_I \alpha$ exists and in fact,

$$y \lim_I \alpha = \lim_I y\alpha$$

$\square$

Corollary. *A category C admits finite limits iff it admits finite products and equalizers of any two morphisms.*

Proof. If $\alpha : I \to C$ is a functor, and I is finite, then both

$$\prod_i \alpha i \quad \text{and} \quad \prod_{s:i\to j} \alpha j$$

are both finite products. Therefore the equalizer of the morphisms t and $u$in the theorem above is the limit of α. Conversely, we see that finite products are finite limits, and since $\mathbf{Eq}_C(f, g)$ is isomorphic to the limit category of some functor $\mathbf{Lim}(\alpha, I, C)$, where I is finite, we have that equalizers exist. $\square$

Corollary. *For small category I, and locally small category C, if C is admits coproducts and coequalizers, then it is cocomplete with respect to I.*

Proof. Suppose C admits coproducts and coequalizers.

To say that C admits coequalizers is to say that C^{op} admits equalizers. To say that C admits coproducts is to say that C^{op} admits products. To say that C admits coproducts is to say that C^{op} admits products. Therefore C^{op} admits limits, which means that C admits colimits. $\square$

6.11.1 Categories with Limits or Colimits

We see that the category of sets, and the category of A-modules have both products and equalizers and therefore have limits.

We see that since the category of sets, and the category of A-modules have both coproducts and coequalizers and therefore have colimits.

6.12 Filtered Category

We desire to generalize the notion of a directed set. Recall that a directed set is a nonempty pre-ordered set (proset) such that any two elements have an upper bound.

A category I is called "filtered" or "filtrant" iff:

1. I is nonempty
2. For any two objects $i, j \in Ob(I)$ there exists $k \in Ob(I)$ and morphisms $i \to k$ and $j \to k$
3. For any two morphisms $f, g : i \to j$ there exists $h : j \to k$ such that $h \circ f = h \circ g$

A category I is called "cofiltered" iff its opposite category is a filtered category. That is:

1. I is nonempty
2. For any two objects $i, j \in Ob(I)$ there exists $k \in Ob(I)$ and morphisms $k \to i$ and $k \to j$
3. For any two morphisms $f, g : j \to i$ there exists $h : k \to j$ such that $f \circ h = g \circ h$

Obviously if $(I, \leq)$ is a directed proset, then its corresponding category $\mathbf{Poset}(I, \leq)$ is a filtered category, and if it is a codirected set, then its corresponding category is a cofiltered category.

When I is filtered, it is common terminology to call a functor $\alpha : I \to C$ a "inductive system", and call $\operatorname{colim}(\alpha)$ an "inductive limit". In this case, a cocone is also called an "inductive cone". It is often the case that one sees these terms used when I is not filtered, but we suggest not doing this to lessen confusion.

It is more common to use the word "filtered" than "filtrant". We also call a directed set a "filtered set".

6.12.1 The Set Colimit of a Filtered Shape

Recall that the colimit of a functor $\alpha : I \to \mathbf{Set}$ is given as

$$L := \frac{\coprod \alpha(i)}{\sim}$$

where $\sim$ denotes the smallest equivalence containing the relation R defined by

$$(x, i)R(y, j) \iff \exists s : i \to j \text{ such that } y = \alpha(s)(x)$$

Proposition 97. *If I is a filtered shape, then for any finite subset S of L, there exists $k \in Ob(I)$ such that S is contained in the image of $\alpha(k)$.*

Proof. Given $x \in S$, take $y_x \in \alpha(i_x)$ such that $x = [(y_x, i)]$, that is, y_x is the representation of the equivalence class x. Then by filteration, take $k \in Ob(I)$ such that we have morphism $s_x : i_x \to k$ for all x. Then $\alpha(s_x)(y_x, i) \in \alpha(k)$, which means that $[\alpha(s_x)(y_x, i)]$ is in the image of $\alpha(k)$. Now by definition of our relation, we immediately have that $(y_x, i) \sim \alpha(s_x)(y_x, i)$ which means that $x = [\alpha(s_x)(y_x, i)]$. $\square$

Proposition 98. *Define $\simeq$ as the relation on $\coprod \alpha(i)$ which satisfies*

$$(x, i) \simeq (y, j)$$

$$\iff$$

$$\exists s : i \to k, \exists t : j \to k \textit{ such that } \alpha(s)(x) = \alpha(t)(y)$$

If I is a filtered category, then the equivalence $\sim$ generated by R coincides with $\simeq$.

Proof. We shall use the notation when we gave the explicit construction of $\sim$.

We easily see that $\simeq$ is contained in $\sim$. To see the converse inclusion, note that since I is filtrant, we have the existance of $\{k_t\}_t$

$$\begin{tikzcd}
 & \mu_1 \arrow[dl, "d_1"'] \arrow[dr, "c_1"] & & \mu_2 \arrow[dl, "d_2"'] \arrow[dr, "c_2"] & \cdots & \mu_n \arrow[dl, "d_n"'] \arrow[dr, "c_n"] & \\
i = a_{0,1} \arrow[dr, "e_1"'] & & a_{1,2} \arrow[dl, "f_1"] \arrow[dr, "e_2"'] & & \cdots \arrow[dl, "f_{n-1}"] \arrow[dr, "e_n"'] & & j = a_{n,n+1} \arrow[dl, "f_n"] \\
 & k_1 & & k_2 & \cdots & k_n &
\end{tikzcd}$$

such that $e_t \circ d_t = f_t \circ c_t$ for all t. This means that $e_i(b_{t-1,t}) = f_i(b_{t,t+1})$. Continuing this construction, we eventually obtain a series of maps which takes $x \in i = a_{0,1}$ and $y \in j = a_{n,n+1}$, which shows that $(x,i) \simeq (y,j)$. $\square$

Proposition 99. *For ring A, if $(I, \leq)$ is a filtered set, and $\alpha : I \to \mathbf{Mod}(A)$ is a functor, denote $s_{i,j}$ as the unique morphism from i to j in I.*

For $x \in \alpha(i)$, $y \in \alpha(j)$, define the relation $\sim$ on the set $\coprod \alpha(i)$ by putting

$$(x,i) \sim (y,j)$$

$$\iff$$

$$\exists k : i \leq k, j \leq k, \alpha(s_{i,k})(x) = \alpha(s_{j,k})(y)$$

Then $\sim$ is an equivalence. Putting

$$M := \frac{\coprod \alpha(i)}{\sim}$$

$$\gamma_i : \alpha(i) \to M$$

$$\gamma_i : x \mapsto [x,i]$$

makes $(M, \{\gamma_i\}_i)$ a limit of α.

Proof. We note that in particular, for $(x,i) \in \coprod \alpha(i)$, and $j \in Ob(I)$, we have $(x,i) \sim (\alpha(s_{i,j})(x), j)$. The converse, however, is not necessarily true.

We immediately see reflexivity and symmetry. For transitivity, if $(x,i) \sim (y,j) \sim (z,k)$, then simply take t such that $i, j, k \leq t$, and use the definition of $\sim$.

Now we show that M is an A- module. Define addition as follows. For $[x,i], [y,j] \in M$, define $[p,k] \in M$ as their addition iff

$$i, j \leq k, p = \alpha(s_{i,k})(x) + \alpha(s_{j,k})(y)$$

We see that this association is uniquely determined irrespective of k. We easily see that it is associative. The identity is $[0_i, i]$ for some $i \in Ob(I)$; recall that I is nonempty. Inverse and associativity is obvious.

The action is defined by

$$c \cdot [x,i] := [cx, i]$$

for $c \in A$. One verifies it to be well defined, for suppose $[x,i] \sim [y,j]$. Then bring this to $\alpha(s_{i,t})(x) = \alpha(s_{j,t})(y)$.

$$[cx, i] = [c \cdot \alpha(s_{i,t})(x), t] = [\alpha(s_{j,t})(cy), t] = [cy, i]$$

The distributivity properties are immediate.

We immediately see that we have a cone due to the commutativity of

$$\begin{tikzcd} & M & \\ \alpha(i) \arrow[ur, "\gamma_i"] \arrow[rr, "\alpha(s_{i,j})"'] & & \alpha(j) \arrow[ul, "\gamma_j"'] \end{tikzcd}$$

Further, if $(Y, \{f_i\}_i)$ is a cone, then define

$$\mu : [x,i] \mapsto f_i(x)$$

and we see that this is in fact well defined, and is the unique A-module homomorphism from M to Y that is desired. $\square$

Corollary. *If I is a small filtered category, and $\alpha : I \to \mathbf{Mod}(A)$ is a functor, then*

$$for \lim \alpha = \lim for \circ \alpha$$

where $for : \mathbf{Mod}(A) \to \mathbf{Set}$ is the forgetful functor.

Proposition 100. *Filtered colimits commute with finite limits in $\mathbf{Set}$. That is, if I is a filtered category and J is a finite category, and $\alpha : I \times J \to \mathbf{Set}$ is a functor, then there exists isomorphism*

$$\lim \mathit{colim}\ \alpha \approx \mathit{colim} \lim \alpha$$

or to be explicit,

$$\lim \lim(\Xi_I \alpha)^{op} \approx \lim(\lim \Xi_J \alpha)^{op}$$

That is, the functor

$$\mathit{colim} : Fct(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with the

Proof. abc $\square$