# The Mathematics of Monoidal Categories in the Context of Topological Quantum Computers

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# Contents

## Praeambulum

This paper focuses on monoidal functors and gives connections to topological quantum computers. We will develop category theory from scratch, and note how we can justify its foundations. We develop fundamental definitions including functors and natural transformations. From this, we define monoidal categories. We define the category of $R$-modules, and the category of tangles, and then show that they are monoidal categories. We discuss submonoidal categories, and show that the category of braids is a submonoidal category of the category of tangles. It then is natural to consider functors which preserve some sort of structure between monoidal categories. We conclude by introducing the concept of a monoidal functor, and describe two examples. Futher topics are then discussed, including the mathematics of anyons and categorical quantum mechanics.

For the development of category theory, primary references are Lang's Algebra [14], Schapira's Algebra and Topology [22] and Kassel's Quantum Groups [12]; the last of which is primarily referenced in the development of monoidal categories. The development of tangles and braids borrows from Kassel [12]. The section on monoidal functors references Heunen and Vicary's Categorical Quantum Mechanics [8].

# 1 Prerequsites

We shall assume a set theoretic axiomatization of mathematics in this paper. We assume the reader has knowledge of the standard axiomatization of ZFC (Zermelo-Frankael, with axiom of choice). We assume the reader has knowledge of functions, and their associated concepts such as injectivity and surjectivity. We assume knowledge of basic operations of sets, such as unions, intersections, products, and concepts including disjointedness and indexes.

We assume standard knowledge of algebra as taught in undergraduate mathematics. In particular we assume the reader knows the definitions of groups, rings, fields, modules, and vector spaces. We assume knowledge of group homomorphisms, ring homomorphisms, and module homomorphisms. We assume knowledge of the definition of quotients of groups, rings, fields, modules, and vector spaces. We assume understanding of what is meant by a topological space and continuous maps and homeomorphisms, and their properties.

### Definitions and notations

The power set of a set $A$ is denoted as $\mathcal{P}(A)$.

We denote $Map(A, B)$ as the set of all maps from $A$ to $B$. We denote the image of $f \in Map(A, B)$ as either $\mathrm{Im}(f)$ or $f[A]$. More generally, given $a \subset A$, write $f[a]$ to denote

$$f[a] := \{f(x) \mid x \in a\}$$

When $a = (x_1, \ldots, x_m)$ is an ordered set, or an $m$-tuple, we will write $f[a]$ or $f(a)$ to denote

$$(f(x_1), \ldots, f(x_m))$$

# 2 Summary of Topological Quantum Computing

The highly multidisciplinary field of Quantum computation is the currently a focus of the technological industry with investments by Google, IBM and Microsoft into developing a viable error free and error corrected quantum computation device. Google and IBM have focused on superconducting processors [2][6], while Microsoft has focused on encoding topological qubits [7].

Quantum computation is significant because it promises to make calculations faster. A primary area of interest is the asymptotic behaviors of certain problems as our input bits get large. Given $n$ bits of input, and a computational problem, computational complexity is defined by $O(f(n))$ for some function $f$, determined by the problem. Computational problems that are achievable in polynomial time are called "Easy", and those which are not are called "Hard" [18, page 45,46]. Quantum supremacy is the paradigm in which hard problems for classical computers are easy for quantum computers. Famously, there are currently no known classical algorithms that solve the prime-factorization problem in polynomial time [18, page 45]. This is the fact on which RSA encryption schemes are based. However Shor [23] exhibits such an algorithm for a quantum computer. There is currently work being done on the creation of quantum-proof encryption schemes.

A major difficulty in the realization of quantum computers is the preventation of noise and error in computation. Qubits denote a quantum state of a certain system, and any environmental interference with the system disturbs the state. A method of computation robust against such error is topological quantum computation. The physical state of a qubit is realized by the arrangement of four points or particles (or in fact, quasiparticles) in two dimensional space; and switching around these particles corresponds to quantum operations on the qubit [13, page 20]. A possible way to realize these particles are by anyons, which we discuss in section 8. A 2020 paper claims direct observation the existence of anyons [17].

Say we switch around two particles in an anti-clockwise manner as in figure 1. Then the

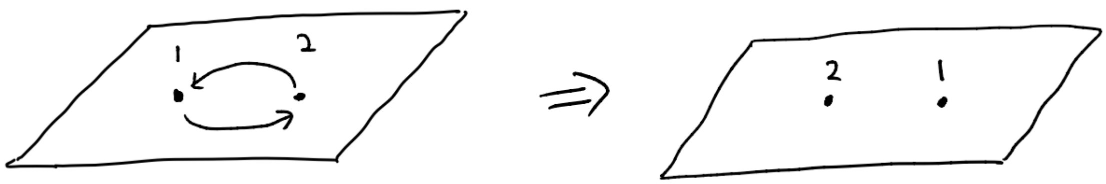


Figure 1: Switching two particles anti-clockwise

state of the system at the end is the same irrespective of the paths we chose to switch those particles. Therefore the operation on the particles is topological in nature, which makes the system less prone to errors. The precision of most quantum states currently achievable is at the order of three nines, that is, 99.9%. The realization of topological quantum systems promises precision at the order of six to seven nines [7], which is a primary motivation for topological quantum computers.

# 3 Category Theory

The current section presents a brief but formal summary of standard definitions and results in category theory, which may be found in [1][12][14][22]. Since this dissertation is not (solely) about category theory, we shall not go too deep in the development, but rather only present prerequisites in understanding and treating the tensor category, which is a central object of study in quantum groups and categorical quantum mechanics.

## 3.1 Categories

Starting from main branches of algebra, such as group theory, ring theory, linear algebra, or module theory, there are concepts of homomorphisms. We wish to understand the concept of a homomorphism more generally. One powerful way to do this is the framework of categories.

The following definition is adopted with consultation from Schapira [22, page 14] and Lang [14, page 53].

**Definition 1.** $\mathbb{A}$ is a "category" iff it is the four-tuple $\mathbb{A} = (\mathfrak{O}, \mathfrak{H}, \mathrm{Hom}, \circ)$ such that the following conditions are satisfied:

1. Hom gives a partition of $\mathfrak{H}$, indexed by the set $\mathfrak{O} \times \mathfrak{O}$. That is to say that $\mathrm{Hom} : \mathfrak{O} \times \mathfrak{O} \to \mathcal{P}(\mathfrak{H})$ is a map such that:

   (a) $\mathfrak{H} = \bigcup_{A.B \in \mathfrak{O}} \mathrm{Hom}(A, B)$, i.e. any element in $\mathfrak{O}$ belongs to $\mathrm{Hom}(A, B)$ for some $A, B \in \mathfrak{O}$. Elements in $\mathrm{Hom}(A, B)$ are called "morphisms from $A$ to $B$", and we write $f : A \to B$ iff $f \in \mathrm{Hom}(A, B)$.

(b) Any two elements in the image of Hom are pairwise disjoint. That is to say, if $\mathrm{Hom}(A, B) = \mathrm{Hom}(A', B')$, then $(A, B) = (A', B')$.

2. For $A, B, C \in \mathfrak{O}$, we impose the following conditions.

   (a) We require that $\circ = (\circ_{A,B,C})_{(A,B,C)\in\mathfrak{O}\times\mathfrak{O}\times\mathfrak{O}}$ is a family of functions indexed by elements in $\mathfrak{O}\times\mathfrak{O}\times\mathfrak{O}$, such that $\circ_{(A,B,C)}$ is a map which associates two morphisms $f : A \to B$ and $g : B \to C$ to some morphism $\circ_{(A,B,C)}(f, g) : A \to C$. As a matter of notation, we write $g \circ f$ to denote $\circ_{(A,B,C)}(f, g)$.

   (b) Given (a) hence, for all $A \in Ob(\mathbb{A})$ there exists an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that for all $B \in Ob(\mathbb{A})$, we have $\forall f \in \mathrm{Hom}(B, A) : 1_A \circ f = f$ and $\forall f \in \mathrm{Hom}(A, B) : f \circ 1_A = f$. It is immediately verified that the identity morphism is unique in the proposition 4.

   (c) For all $A, B, C, D \in Ob(\mathbb{A})$, and for all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$, we have $(f \circ g) \circ h = f \circ (g \circ h)$. (Note that if 2. (a) holds, then both sides of the equality must exist; that is, we have that both $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are in $\mathrm{Hom}(A, D)$, and are equal; the equality is in fact always, "defined"). This is the condition of associativity.

*Remark* 2. Certain sources such as [14, page 53] may choose to ignore condition 1 (b). Call such a 4-tuple that satisfies such a definition a pre-category. Any pre-category can be turned into a category, as declared in lemma 3. This is illuminated in the category of sets, which we show to be a category in example 6. Bergman [3, page 34] discusses this further.

**Lemma 3.** *Any pre-category* $(\mathfrak{O}, \mathfrak{H}, \mathrm{Hom}, \circ)$ *can be turned into a category as follows.*

*Given objects* $A, B$, *and morphism* $f \in \mathrm{Hom}(A, B)$, *set*

$$\overline{\mathrm{Hom}}(A, B) := \{(A, f, B) \mid f \in \mathrm{Hom}(A, B)\}$$

$$\overline{\mathfrak{H}} := \{(A, f, B) \mid f \in \overline{\mathfrak{H}}\}$$

*and given morphisms* $f : A \to B$ *and* $g : B \to C$, *put*

$$(A, f, B)\bar{\circ}(B, g, C) := (A, g \circ f, C)$$

*Then this* $(\mathfrak{O}, \overline{\mathfrak{H}}, \overline{\mathrm{Hom}}, \bar{\circ})$ *is a category.*

It only requires to check the above conditions to see that this in fact defines a category, so we will not give details of the proof. A morphism $f$ from $A$ to $B$ is therefore treated synonymously as $(A, f, B)$.

**Proposition 4.** *For any object* $A$ *in category* $\mathbb{A}$, *its identity morphism is unique.*

*Proof.* Suppose $1_A, 1'_A \in \mathrm{Hom}(A, A)$ are both identity morphisms of object $A$. Then from the fact that $1_A$ is the identity morphism, have $\forall f \in \mathrm{Hom}(B, A) : 1_A \circ f = f$. From this we obtain $1_A \circ 1'_A = 1'_A$ by substituting $1'_A$ for $f$. From the fact that $1'_A$ is the identity morphism we get $\forall f \in \mathrm{Hom}(A, B) : f \circ 1'_A = f$. Therefore we have $1_A \circ 1'_A = 1_A$ by substituting $1_A$ for $f$. Therefore $1_A = 1_A \circ 1'_A = 1'_A$. So the identity morphism is unique. □

Therefore instead of saying "an identity morphism" we instead say "the identity morphism".

Given category $\mathbb{A} = (\mathfrak{O}, \mathfrak{H}, \text{Hom}, \circ)$, we will denote $\mathfrak{O}$ as $Ob(\mathbb{A})$, and $\mathfrak{H}$ as $\text{Hom}(\mathbb{A})$. We call $Ob(\mathbb{A})$ the set of all objects of $\mathbb{A}$, and we call $\text{Hom}(\mathbb{A})$ the set of morphisms of $\mathbb{A}$.

The first example that we give of a category is that of sets. We wish to say that sets are objects of this category, and maps between them are morphisms. Unfortunately, the set of all sets does not exist. There are ways to circumvent this, for example by proper classes, which declares certain objects as things that are distinct from sets, which can properly contain sets. NBG is a way to axiomatize such a theory of classes. There are also other axiomatizations, such as SEAR or ETCS, which we will not describe. We here state the Bourbakian method of universes, mentioned by Bergman [3, page 36].

**Definition 5.** [4] A set $\mathcal{U}$ is called a "Grothendieck Universe" iff it satisfies the following properties:

1. $x \in \mathcal{U}$ and $y \in x$ implies that $y \in \mathcal{U}$.
2. $x, y \in \mathcal{U}$ implies $\{x, y\} \in \mathcal{U}$.
3. $x \in \mathcal{U}$ implies $\mathcal{P}(x) \in \mathcal{U}$.
4. For all $I \in \mathcal{U}$ and functions $u : I \to \mathcal{U}$, the union of the image of $u$ is in $\mathcal{U}$.

The axiom $U$ states that for any set $S$, there exists a Grothendieck Universe $\mathcal{U}$ such that $S \in \mathcal{U}$.

[15, page 3] proves that any Grothendieck Universe $\mathcal{U}$ is a model of the theory ZFC. From now on, fix $\mathcal{U}$ to be some universe that contains the empty set. How this allows us to define the category of sets is illuminated in example 6.

**Example 6.** The category of sets is denoted as **Set**. Recall that we affixed some universe $\mathcal{U}$ such that $\varnothing \in \mathcal{U}$. We shall call all elements in $\mathcal{U}$ as "$\mathcal{U}$-small sets", or simply, "small sets". These are the objects of **Set**.

For $A, B \in \mathcal{U}$, denote

$$\text{Hom}(A, B) := \{(A, f, B) \mid f \in Map(A, B)\}$$

Denote

$$\mathfrak{H} := \{x \in \mathcal{U} \mid \exists A, B \in \mathcal{U} : x = \text{Hom}(A, B)\} = \bigcup_{A, B \in \mathcal{U}} \text{Hom}(A, B)$$

Then the association Hom which takes $A, B \in \mathcal{U}$ to $\text{Hom}(A, B)$ is a map from $\mathcal{U} \times \mathcal{U}$ to $\mathcal{P}(\mathfrak{H})$.

Then for $A, B, C \in \mathcal{U}$, and sets $\text{Hom}(A, B), \text{Hom}(B, C)$, denote $\circledcirc_{(A,B,C)}$ as the association from $\text{Hom}(A, B) \times \text{Hom}(B, C)$ to $\text{Hom}(A, B)$ which takes

$$\circledcirc_{(A,B,C)} : ((A, f, B), (B, g, C)) \mapsto (A, g \circ f, C)$$

where $f \circ g$ is the usual composition of maps.

Then for $A, B, C \in \mathcal{U}$, denote $\circledcirc$ as the map which takes $A, B, C \in \mathcal{U}$ and maps it to $\circledcirc_{(A,B,C)}$.

**Proposition 7.** *The 4-tuple* $(\mathcal{U}, \mathfrak{H}, Hom, \circledcirc)$ *is a category.*

*Proof.* We continue to verify the conditions.

1. Hom is indeed a map from $\mathcal{U} \times \mathcal{U}$ to $\mathcal{P}(\mathfrak{H})$, because it takes $(A, B)$ to $\mathrm{Hom}(A, B) \subset \mathfrak{H}$.

   (a) By definition, the union of all elements of the image of Hom is precisely $\mathfrak{H}$.

   (b) Suppose $(A, B) \neq (A', B')$. Suppose $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') \neq \varnothing$. Take an element in $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B')$. Then it is expressed as a 3 tuple $(x, y, z)$, where $x = A$, $z = B$, and $y$ is some function from $A$ to $B$. But then we also have $x = A', z = B'$, which means $(A, B) = (A', B')$, a contradiction.

2. Suppose $A, B, C \in Ob(\mathcal{U})$.

   (a) Then we have $\circledcirc_{A,B,C}$ is a map from $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C)$ to $\mathrm{Hom}(A, C)$ by definition.

   Given set $A \in \mathcal{U}$, Denote $\mathrm{id}_A$ as the identity map which takes $a \in A$ to $a \in A$. Then $(A, \mathrm{id}_A, A)$ is a morphism of $A$; that is, $(A, \mathrm{id}_A, A) \in \mathrm{Hom}(A, A)$. Now suppose $B \in \mathcal{U}$, and $(B, f, A) \in \mathrm{Hom}(B, A)$. Then we have that $\circledcirc_{B,A,A}((B, f, A), (A, \mathrm{id}_A, A)) = (B, \mathrm{id}_A \circ f, A)$. On the other hand, suppose $(A, f, B) \in \mathrm{Hom}(A, B)$. Then we have that $\circledcirc_{A,A,B}((A, \mathrm{id}_A, A), (A, f, B)) = (A, f \circ \mathrm{id}_A, B)$. So:

   (b) For all $B \in \mathcal{U}$, we have $\forall \gamma \in \mathrm{Hom}(B, A) : 1_A \circ \gamma = \gamma$ and $\forall \gamma \in \mathrm{Hom}(A, B) : \gamma \circ 1_A = \gamma$, and $1_A = (A, \mathrm{id}_A, A)$.

   (c) Suppose $A, B, C, D \in \mathcal{U}$, for all $(A, f, B) \in \mathrm{Hom}(A, B)$, $(B, g, C) \in \mathrm{Hom}(B, C)$, and $(C, h, D) \in \mathrm{Hom}(C, D)$. Then

   $$((A, f, B) \circledcirc (B, g, C)) \circledcirc (C, h, D) = (A, g \circ f, C) \circledcirc (C, h, D) = (A, h \circ (g \circ f), D)$$

   $$(A, f, B) \circledcirc ((B, g, C) \circledcirc (C, h, D)) = (A, f, B) \circledcirc (B, h \circ g, D) = (A, (h \circ g) \circ f), D)$$

   Observing that $(A, h \circ (g \circ f), D) = (A, (h \circ g) \circ f), D)$, we conclude that for all $\gamma \in \mathrm{Hom}(A, B)$, $\eta \in \mathrm{Hom}(B, C)$, and $\mu \in \mathrm{Hom}(C, D)$, we have $(\gamma \circ \eta) \circ \mu = \gamma \circ (\eta \circ \mu)$.

So all the conditions are verified. ☐

*Remark* 8. The condition that given any two Hom sets, we need $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ to be disjoint whenever $(A, B) \neq (A', B')$ is required if we want to distinguish cases when we want the codomain of a function to be well defined in the category of sets.

It is, however, possible to write $f$ to denote $(A, f, B)$ for notational brevity, as long as we understand that we also need to include the domain and codomain information in the morphism.

**Example 9.** The next proposition shows how we can turn a monoid into a category.

**Proposition 10.** *For monoid* $(M, *)$*, denote* $\mathbf{Cat}(M, *)$ *as the 4-tuple*

$$(\{\varnothing\}, M, \mathrm{Hom}, \{(\circ_{\varnothing,\varnothing,\varnothing})\})$$

*where:*

*1.* $\mathrm{Hom} : \{\varnothing\} \times \{\varnothing\} \to \{M\}$ *is the unique possible map.*

*2.* $\circ_{\varnothing,\varnothing,\varnothing} = *$.

*Then* $\mathbf{Cat}(M, *)$ *is a category.*

*Proof.* We verify each condition.

1. Since $\mathrm{Hom} : \{\varnothing\}\times\{\varnothing\} \to \{M\}$ is a map, we automatically see that it takes $(\varnothing, \varnothing) \mapsto M$. So it is in fact a partition of $M$, of one part only.

2. Suppose $A, B, C, D \in \{\varnothing\}$. Then $A = B = C = D = \varnothing$.

   (a) We have that $\{(\circ_{\varnothing,\varnothing,\varnothing})\}$ is an indexed family, with index belonging to $\{\varnothing\} \times \{\varnothing\} \times \{\varnothing\}$. We have $\mathrm{Hom}(\varnothing, \varnothing) = M$. Since we have $* : M \times M \to M$. That is, $* : \mathrm{Hom}(\varnothing, \varnothing) \times \mathrm{Hom}(\varnothing, \varnothing) \to \mathrm{Hom}(\varnothing, \varnothing)$. That is, $\circ_{\varnothing,\varnothing,\varnothing} \in M_{\varnothing,\varnothing,\varnothing}$.

   (b) We have that the identity $e_M \in M = \mathrm{Hom}(\varnothing, \varnothing)$. Given morphism $f : \varnothing \to \varnothing$, we have that $f \circ e_M = f * e_M = f$. Also given $g : \varnothing \to \varnothing$, we have that $e_M \circ g = e_M * g = g$. So $e_M$ is the identity in $\mathrm{Hom}(\varnothing, \varnothing)$.

   (c) Given $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, we have

   $$f, g, h \in \mathrm{Hom}(\varnothing, \varnothing) = M$$

   So $f, g, h \in M$. So

   $$(f \circ g) \circ h = (f * g) * h = f * (g * h) = f \circ (g \circ h)$$

□

Further well known examples of categories are noted, but we do not go into detail.

**Example 11.** Rings, groups, $R$-modules, $k$-vector spaces, topological spaces, pointed topological spaces, and are all pre-categories, which can be turned into categories. The category of (left) $R$-modules is denoted $\mathbf{Mod}(R)$ or $R\mathbf{Mod}$. The category of all vector spaces over field $k$ is denoted $\mathbf{Vect}_k$. The category of all topological spaces is denoted **Top**. The category of all pointed topological spaces is denoted $\mathbf{Top}_\bullet$. The category of all rings is denoted **Ring**. The cateogry of all groups is denoted **Grp**. The category of all affine spaces over field $k$ is denoted $\mathbf{Affine}_k$.

**Example 12.** The empty category, which has no objects and hence no morphisms is a category. The terminal category is the category that has exactly one object, and one morphism: the identity from the object to itself. The terminal category is often denoted as $\{\bullet\}$ or as pt.

**Example 13.** It shall be shown that tangles and braids form categories.

Given element $f \in \mathrm{Hom}(\mathbb{A})$, we will write, as a logical statement $f : A \to B$ (or $A \xrightarrow{f} B$) iff $f \in \mathrm{Hom}(A, B)$. In this case we will say that $A$ is the source of $f$ and $B$ is the target of $f$. This is compactly denoted by writing $\sigma(f) = A$ and $\tau(f) = B$.

It is easy to see that whenever $A \xrightarrow{f} B \xrightarrow{g} C$, we have that $g \circ f$ exists in the category. We say that $f \circ g$ is the “composition” of $f$ and $g$. Morphisms, however, in general do not need to be functions or maps.

Given morphism $f : A \to B$ and morphism $g : C \to B$ we will state that "$g$ is composible with $f$ on the left" or "$f$ is composible with $g$ on the right" iff $B = C$, that is, the source of $g$ coincides with the target of $f$. When this is true, the composition $\circ_{(A,B,C)}(f, g) = g \circ f$ is defined.

A possibly infinite set of morphisms from a category is called a "diagram". A diagram is said to commute iff given any two compositions of morphisms in the diagram $f_1 \circ f_2 \circ \cdots \circ f_n$ and $g_1 \circ g_2 \circ \cdots \circ g_m$, if they both have the same source and target, then they are equal.

**Example 14.** To say that the diagram given in figure 2 commutes is to say that $g \circ f = g' \circ f'$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & C \arrow[d, "g"] \\ B \arrow[r, "g'"'] & D \end{tikzcd}$$

Figure 2: Example of a diagram.

**Definition 15.** An element in $Ob(\mathbb{A})$ is called an "object of $\mathbb{A}$". We denote the identity morphism of object $X$ by $1_X$ or $\mathrm{id}_X$.

**Example 16.** Any subset of $\mathcal{U}$ is an object of **Set** by definition, as given in example 6.

**Example 17.** The identity map from $A$ to itself gives the identity morphism of $A$ in **Set**.

**Definition 18.** In the context of some category $\mathbb{A}$, we say that a set $S$ is a "Homset" or "Hom-set" iff there exist two objects $A, B \in Ob(\mathbb{A})$ such that $S = \mathrm{Hom}(A, B)$.

**Definition 19.** Given a category $(Ob(\mathbb{A}), \mathrm{Hom}(\mathbb{A}), \mathrm{Hom}, \circ)$, the opposite category is the four tuple $(Ob(\mathbb{A}^{\mathrm{op}}), \mathrm{Hom}(\mathbb{A}^{\mathrm{op}}), \mathrm{Hom}_{\mathrm{op}}, \circ_{\mathrm{op}})$ where

1. $Ob(\mathbb{A}^{\mathrm{op}}) = Ob(\mathbb{A})$, and $\mathrm{Hom}(\mathbb{A}^{\mathrm{op}}) = \mathrm{Hom}(\mathbb{A})$.

2. $\mathrm{Hom}_{\mathrm{op}}$ maps from $Ob(\mathbb{A}^{\mathrm{op}}) \times Ob(\mathbb{A}^{\mathrm{op}}) \to \mathcal{P}(\mathrm{Hom}(\mathbb{A}^{\mathrm{op}}))$ such that given any two objects $A, B \in Ob(\mathbb{A}^{\mathrm{op}})$, we have that $\mathrm{Hom}_{\mathrm{op}}(A, B) = \mathrm{Hom}_{\mathrm{op}}(B, A)$.

3. $(\circ_{\mathrm{op}A,B,C})_{A,B,C \in Ob(\mathbb{A})}$ is an indexed set, where $\circ_{\mathrm{op}A,B,C}$ is a map which takes $\mathrm{Hom}_{\mathrm{op}}(A, B) \times \mathrm{Hom}_{\mathrm{op}}(B, C) \to \mathrm{Hom}_{\mathrm{op}}(A, C)$ so that

$$\circ_{\mathrm{op}A,B,C} : (f, g) \mapsto \circ_{C,B,A}(g, f)$$

Note that $\circ_{C,B,A}(g, f)$ is defined.

**Proposition 20.** *The opposite category is a category.*

We omit the proof of the above proposition as it only requires to check the conditions stated in definition 1.

## 3.2 Morphisms and Isomorphisms

A particular class of morphisms of crucial importance are isomorphisms.

**Definition 21.** For morphism $f \in \mathrm{Hom}(A, B)$, when $g \in \mathrm{Hom}(B, A)$ and $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, we say that "$g$ is an inverse of $f$." If an inverse of $f$ exists, then $f$ is said to be an "isomorphism". A morphism that is an isomorphism is also called "invertible".

**Example 22.** A morphism $(A, f, B)$ in the category of sets is an isomorphism iff $f$ is a bijection.

**Proposition 23.** *The identity morphism of any object is an isomorphism.*

*Proof.* For object $A$, we have that $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. □

**Proposition 24.** *The inverse of an isomorphism is unique.*

*Proof.* If $g, h$ are inverses of $f$, then $g = g \circ (f \circ h) = (g \circ f) \circ h = h$. □

**Proposition 25.** *The inverse of an isomorphism is an isomorphism.*

*Proof.* Immediate from definition of an isomorphism. □

We shall denote the inverse of an isomorphism $f$ as $f^{-1}$.

**Definition 26.** For category $\mathbb{A}$ and two objects $A, B \in Ob(\mathbb{A})$, we state that "$A$ is isomorphic to $B$" iff there exists some morphism $f : A \to B$ such that $f$ is an isomorphism.

**Example 27.** Given the category of sets, we have that any two sets which have a bjiection between them are isomorphic.

**Proposition 28.** *If $A$ is isomorphic to $B$, then $B$ is isomorphic to $A$.*

*Proof.* Follows from proposition 25. □

**Proposition 29.** *The relation $\sim$ which says that $A \sim B$ iff $A$ is isomorphic to $B$ is an equivalence relation.*

*Proof.* We have $A \sim A$ by the identity $\mathrm{id}_A$. We have that if $A \sim B$ by an isomorphism $f : A \to B$, then $B \sim A$ by $f^{-1} : B \to A$.

Finally suppose $A \sim B$ and $B \sim C$. Then take $f : A \to B$ and $g : B \to C$. We have that $g \circ f : A \to C$. Take the inverses of $f$ and $g$ and denote them as $f^{-1}$ and $g^{-1}$ respectively. We have that $f^{-1} : B \to A$ and $g^{-1} : C \to B$. So the are composible as $f^{-1} \circ g^{-1} : C \to A$. We get that

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ (f^{-1} \circ g^{-1})) \\ &= g \circ ((f \circ f^{-1}) \circ g^{-1}) \\ &= g \circ (\mathrm{id}_B \circ g^{-1}) \\ &= g \circ g^{-1} \\ &= \mathrm{id}_C\end{aligned}$$

and similarly, we obtain

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_A$$

and therefore $(g \circ f)$ is an isomorphism, whose source is $A$ and target is $C$. We thus have $A \sim C$. □

### 3.3 Product of Categories

Lending intuition from how we define, for example the product of groups, the product of vector spaces, or the product of rings, it is clear how we can define the product of two categories.

**Definition 30.** Given indexed set of categories $(\mathbb{A}_i)_{i\in I}$, where $\mathbb{A}_i$ denotes a four-tuple

$$(Ob(\mathbb{A}_i), \mathrm{Hom}(\mathbb{A}_i), \mathrm{Hom}_i, \circ_i)$$

we shall say that a four tuple $(\mathfrak{O}_{\prod}, \mathfrak{H}_{\prod}, \mathrm{Hom}_{\prod}, \circ_{\prod})$ is the "categorical product of $(\mathbb{A}_i)_{i\in I}$" iff:

1. We have the set equality $\mathfrak{O}_{\prod} = \prod Ob(\mathbb{A}_i)$
2. We have the set equality $\mathfrak{H}_{\prod} = \prod \mathrm{Hom}(\mathbb{A}_i)$
3. $\mathrm{Hom}_{\prod}$ is a function from $\mathfrak{O}_{\prod} \times \mathfrak{O}_{\prod}$ to $\mathfrak{H}_{\prod}$ such that for any two elements $(A_i)_i, (B_i)_i \in \prod Ob(\mathbb{A}_i)$, we have
$$\mathrm{Hom}_{\prod}((A_i)_i, (B_i)_i) = \prod_i \mathrm{Hom}_i(A_i, B_i)$$
4. $\circ_{\prod (A_i)_i,(B_i)_i(C_i)_i}$ is a map which takes
$$\mathrm{Hom}((A_i)_i, (B_i)_i) \times \mathrm{Hom}((B_i)_i, (C_i)_i) \to \mathrm{Hom}((B_i)_i, (C_i)_i)$$
such that given two morphisms $(\alpha_i)_i : (A_i)_i \to (B_i)_i$, $(\beta_i)_i : (B_i)_i \to (C_i)_i$, we have $(\beta)_i \circ_{\prod} (\alpha_i)_i := (\beta_i \circ_i \alpha_i)_i$.

Then it is routinely verified, by checking the definition, that this obtains a category, as proved in the proposition below.

**Proposition 31.** *The categorical product of an indexed set of categories $(\mathbb{A}_i)_i$ is a category.*

*Proof.* We want to show that $(\mathfrak{O}_{\prod}, \mathfrak{H}_{\prod}, \mathrm{Hom}_{\prod}, \circ_{\prod})$ satisfies the conditions in our definition.

1. We want to show $\mathrm{Hom}_{\prod}$ gives a partition of $\mathfrak{H}_{\prod}$, indexed by the set $\mathfrak{O}_{\prod} \times \mathfrak{O}_{\prod}$.
   (a) Suppose that $(\alpha_i)_i \in \prod \mathrm{Hom}(\mathbb{A}_i)$. Then for each $i \in I$, take $A_i, B_i \in Ob(\mathbb{A}_i)$ such that $\alpha_i \in \mathrm{Hom}(A_i, B_i)$. Then we have that for such an index of objects, we have $(A_i)_i \in Ob_{\prod}(\mathbb{P})$ and $(B_i)_i \in Ob_{\prod}(\mathbb{P})$. By 3. of the above definition, we have $\mathrm{Hom}_{\prod}((A_i)_i, (B_i)_i) = \prod \mathrm{Hom}_i(A_i, B_i)$, and so $(\alpha_i)_i \in \mathrm{Hom}_{\prod}((A_i)_i, (B_i)_i)$. So $\mathrm{Hom}_{\prod}$ is a cover of $\mathfrak{H}_{\prod}$.
   (b) Suppose that $\mathrm{Hom}_{\prod}((A_i)_i, (B_i)_i) \cap \mathrm{Hom}_{\prod}((A'_i)_i, (B'_i)_i)$ is non-empty. Then take an element $(\alpha_i)_i$ in that set. It is in $\prod \mathrm{Hom}_{\mathbb{A}_i}(A_i, B_i)$, which means that for arbitrary $i$, $\alpha_i \in \mathrm{Hom}_i(A_i, B_i)$. Further, it is in $\prod \mathrm{Hom}_i(A'_i, B'_i)$, which means that $\alpha_i \in \mathrm{Hom}_i(A'_i, B'_i)$. So $\mathrm{Hom}_i(A_i, B_i) \cap \mathrm{Hom}_i(A'_i, B'_i)$ is non-empty, so $A_i = A'_i$ and $B_i = B'_i$. This holds for all $i$, so we have that $(A_i)_i = (A'_i)_i$ and $(B_i)_i = (B'_i)_i$.
2. Suppose $(A_i)_i, (B_i)_i, (C_i)_i, (D_i)_i \in \mathfrak{O}_{\prod}$, and suppose $(f_i)_i : (A_i)_i \to (B_i)_i$, $(g_i)_i : (B_i)_i \to (C_i)_i$, and $(h_i)_i : (C_i)_i \to (D_i)_i$.
   (a) If $(f_i)_i : (A_i)_i \to (B_i)_i$ and $(g_i)_i : (B_i)_i \to (C_i)_i$, then for each $i$, we have $f_i : A_i \to B_i$ and $g : B_i \to C_i$ in $\mathbb{A}_i$, and therefore $g_i \circ f_i : A_i \to B_i$ in $\mathbb{A}_i$. So $(g_i \circ f_i)_i$ is an element of $\mathrm{Hom}_{\prod}((A_i)_i, (C_i)_i)$.

(b) Given $i \in I$, take each identity element $1_i \in Ob(\mathbb{A}_i)$. Then $(1_i)_i \in \mathrm{Hom}_{\prod}((A_i)_i, (A_i)_i)$ is the identity.

(c) We have

$$\begin{aligned}((f_i)_i \circ_{\prod} (g_i)_i) \circ_{\prod} (h_i)_i &= ((f_i \circ_i g_i)_i) \circ_{\prod} (h_i)_i \\ &= ((f_i \circ_i g_i) \circ_i h_i)_i \\ &= (f_i \circ_i (g_i \circ_i h_i))_i \\ &= (f_i)_i \circ_{\prod} (g_i \circ_i h_i)_i \\ &= (f_i)_i \circ_{\prod} ((g_i)_i \circ_{\prod} (h_i)_i)\end{aligned}$$

So we have a category. □

This allows us to define the product of three categories $\mathbb{A} \times \mathbb{A}' \times \mathbb{A}''$ without concern as to which order we take the product. This will be necessary for our discussion on the tensor products of categories. In particular we refer to example 47 to see a usage of the product of two categories.

**Lemma 32.** *For categories $\mathbb{A}, \mathbb{B}$, and their product, $\mathbb{A} \times \mathbb{B}$, given morphism $(f, g) : (A, B) \to (A', B')$ in $\mathbb{A} \times \mathbb{B}$, we have $(f, g) = (f, id_{B'}) \circ (id_A, g) = (id_{A'}, g) \circ (f, id_B)$.*

*Proof.* We have $\mathrm{id}_A : A \to A$ and $f : A \to A'$, so we can compose them as $f \circ \mathrm{id}_A = f$. Also, we have $g : B \to B'$ and $\mathrm{id}_{B'} : B' \to B'$, so we can compose them as $\mathrm{id}_{B'} \circ g = g$. Therefore we get

$$(f, \mathrm{id}_{B'}) \circ (\mathrm{id}_A, g) = (f \circ \mathrm{id}_A, \mathrm{id}_{B'} \circ g) = (f, g)$$

On the other hand, we have $f : A \to A'$ and $\mathrm{id}_{A'} : A' \to A'$, so we can compose them as $\mathrm{id}_{A'} \circ f = f$. Also, we have $\mathrm{id}_B : B \to B$ and $g : B \to B'$, so we can compose them as $g \circ \mathrm{id}_B = g$. Therefore we get

$$(\mathrm{id}_{A'}, g) \circ (f, \mathrm{id}_B) = (\mathrm{id}_{A'} \circ f, g \circ \mathrm{id}_B) = (f, g)$$

□

### 3.4 Terminal and Initial Objects (Universal Objects)

**Definition 33.** [14, page 57] For an object $A$ in a category:

- $A$ is called "universally repelling" or "initial" iff: for all objects $B$ the set $\mathrm{Hom}(A, B)$ is a singleton.
- $A$ is called "universally attracting" or "terminal" or "final" iff: for all objects $B$ the set $\mathrm{Hom}(B, A)$ is a singleton.
- $A$ is called "zero" iff it is both inital and terminal.

It is to be noted here that the usage of the word "universal" has no connection to the Grothendieck Universe.

**Example 34.** The empty set is an example of an initial object in the category of sets. Singleton sets are terminal in the category of sets. The trivial group is both terminal and initial in the category of groups.

**Proposition 35.** *Universally repelling or attracting objects are unique up to isomorphism. Further, isomorphisms between universal objects are unique.*

*Proof.* Suppose $X$ and $Y$ are terminal in category $\mathbb{A}$. Then there exist exactly one object in $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(Y, X)$. Take such morphisms

$$f : X \to Y$$

$$g : Y \to X$$

We have that

$$g \circ f : X \to X$$

$$f \circ g : Y \to Y'$$

Then we have that since $X$ and $Y$ are terminal, we have that there exist exactly one object in $\mathrm{Hom}(X, X)$ and $\mathrm{Hom}(Y, Y)$. But we have

$$\mathrm{id}_X : X \to X$$

$$\mathrm{id}_Y : Y \to Y$$

And therefore $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. That is, $f$ and $g$ are isomorphisms, and so $X$ is isomorphic to $Y$.

Now suppose that $f : X \to Y$ and $g : X \to Y$ is an isomorphism between terminal objects. Since $Y$ is terminal, there exists exactly one morphism in $\mathrm{Hom}(X, Y)$. Therefore $f = g$.

A similar proof holds for initial objects. Alternatively we can observe that it follows from the fact that initial objects are terminal in the opposite category. □

### 3.5 Functors

We now define the notion of functors, which allow us to make connections between two categories.

**Definition 36.** [14, page 62] [22, page 16, 17] Given two categories $\mathbb{A}$ and $\mathbb{B}$, a covariant functor from $\mathbb{A}$ to $\mathbb{B}$ is an ordered pair of maps $(F, \overline{F})$, such that $F : Ob(\mathbb{A}) \to Ob(\mathbb{B})$ and $\overline{F} : \mathrm{Hom}(\mathbb{A}) \to \mathrm{Hom}(\mathbb{B})$, such that:

1. If $f : A \to B$ in $\mathbb{A}$, then $\overline{F}(f) : F(A) \to F(B)$ in $\mathbb{B}$.
2. For identity $\mathrm{id}_A : X \to X$ in $\mathbb{A}$, we have that $\overline{F}(\mathrm{id}_A)$ is the identity of $F(A)$.
3. Given any two composible morphisms $f, g \in \mathrm{Hom}(\mathbb{A})$, we have that $\overline{F}(g \circ_{\mathbb{A}} f) = \overline{F}(g) \circ_{\mathbb{B}} \overline{F}(f)$.

**Example 37.** The identity functor of a category $\mathbb{A}$, denoted $\mathrm{id}_{\mathbb{A}}$ is the pair of associations

$$A \mapsto A$$

$$f \mapsto f$$

for any object $A$ and any morphism $f$.

**Example 38.** We have noted that groups can be viewed as categories in example 10. Let $\mathbb{A}$ be a category, and let $(G, \cdot)$ be a group. Denote **Cat**(G,·) as the category associated with $(G, \cdot)$. A functor $F$ from **Cat**(G,·) to $\mathbb{A}$ is said to be a "group representation of $G$ in $\mathbb{A}$". The object in **Cat**(G,·) is taken to some object $A$ in $\mathbb{A}$, and any morphism in $G$ is taken to some morphism in $\mathrm{Hom}(A, A)$. In fact, since functors preserve isomorphisms we have that it must be in $\mathrm{Aut}(A)$. In particular, when $\mathbb{A} = \mathbf{Vect}_k$, and $A = V$ for some vector space $V$, we have that $\mathrm{Aut}(V) = \mathrm{GL}(V) \approx \mathrm{GL}(n, k)$, where $n$ is the dimension of $V$.

**Example 39.** In my undergraduate thesis [21, page 38], I mention a functor from the category of affine algebraic varieties over field $k$, to the category of rings. [16, page 334] gives an example of a functor from $\mathbf{Top}_\bullet$ to **Grp**.

The dual notion of a covariant functor is the contravariant functor. We will not describe it here, however, it is noted that the definition of dagger categories makes use of this definition (see section 8 for an overview).

In this paper, and in convention, we shall call covariant functors as simply "functors". In a slight abuse of notation, we shall use a single symbol $F$ as both as the map between objects and the map between morphisms, as is conventional to do so. Therefore, we will write $F(f) : F(X) \to F(Y)$, for example.

**Proposition 40.** *If $\mathbb{A}$ and $\mathbb{B}$ are two categories, and $F$ is a functor from $\mathbb{A}$ and $\mathbb{B}$, then for any morphism $f$ such that $f$ is an isomorphism in $\mathbb{A}$, we have that $F(f)$ is an isomorphism in $\mathbb{B}$. That is to say, functors preserve isomorphisms. Furthermore, the inverse of isomorphism $F(f)$ is given by $F(f^{-1})$.*

*Proof.* Suppose that $f : A \to B$ is an isomorphism. Take $f^{-1} : B \to A$ as its inverse. Then we have that $F(f) : F(A) \to F(B)$ and

$$\mathrm{id}_{F(A)} = F(\mathrm{id}_A) = F(f^{-1} \circ f) = F(f^{-1}) \circ F(f)$$

$$\mathrm{id}_{F(B)} = F(\mathrm{id}_B) = F(f \circ f^{-1}) = F(f) \circ F(f^{-1})$$

So $F(f)$ is an isomorphism, and $F(f^{-1})$ is its inverse. □

### 3.6 Composition of Functors

It happens that we can compose functors. The composition of functors is a functor, just as compositions of homomorphisms is a homomorphism.

**Definition 41.** Given three categories $I$, $J$, $K$, and functors $(F, \overline{F}) : I \to J$, $(G, \overline{G}) : J \to K$, we have the maps

$$F : Ob(I) \to Ob(J)$$

$$\overline{F} : \mathrm{Hom}(I) \to \mathrm{Hom}(J)$$

$$G : Ob(J) \to Ob(K)$$

$$\overline{G} : \mathrm{Hom}(J) \to \mathrm{Hom}(K)$$

Then the composition of maps

$$G \circ F : Ob(I) \to Ob(K)$$

$$\overline{G} \circ \overline{F} : \mathrm{Hom}(I) \to \mathrm{Hom}(K)$$

we shall denote the composition of functors $(G, \overline{G}) \circ (F, \overline{F})$ as the pair of associations $(G \circ F, \overline{G} \circ \overline{F})$. Then the composition of functors is in fact a functor, as proven below.

**Proposition 42.** *$(G \circ F, \overline{G} \circ \overline{F})$, as defined above, is a functor from $I$ to $K$.*

*Proof.* We already have that the pair associates objects to objects and morphisms to morphisms from $I$ to $K$. We check the three conditions.

1. Suppose $f : A \to B$ in $I$, then $\overline{F}(f) : F(A) \to F(B)$ in $J$ and $\overline{G}(\overline{F}(f)) : G(F(A)) \to G(F(A))$. That is, $(\overline{G} \circ \overline{F})(f)) : (G \circ F)(A) \to (G \circ F)(A)$
2. For identity $\mathrm{id}_A : X \to X$ in $I$, we have that $\overline{F}(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, and so $\overline{G}(\overline{F}(\mathrm{id}_A)) = \mathrm{id}_{G(F(A))}$. That is, $(\overline{G} \circ \overline{F})(\mathrm{id}_A) = \mathrm{id}_{(G \circ F)(A)}$.
3. Given any two composible morphisms $f, g \in \mathrm{Hom}(I)$, we have that

$$\begin{aligned}
(\overline{G} \circ \overline{F})(g \circ_I f) &= \overline{G}(\overline{F}(g \circ_I f)) \\
&= \overline{G}(\overline{F}(g) \circ_J \overline{F}(f)) \\
&= \overline{G}(\overline{F}(g)) \circ_K \overline{G}(\overline{F}(f)) \\
&= (\overline{G} \circ \overline{F})(g) \circ_K (\overline{G} \circ \overline{F})(f)
\end{aligned}$$

When denoting functors $(F, \overline{F})$ as simply $F$, we denote the composition of functors $F$ and $G$ as $G \circ F$. □

## 3.7 Product of Functors

**Definition 43.** Given an indexed set $I$ and two indexed set of categories $(\mathbb{A}_i)_{i \in I}$ and $(\mathbb{B}_i)_{i \in I}$, and indexed set of functors $(F_i)_{i \in I}$ such that for each $i \in I$, we have that $F_i : \mathbb{A}_i \to \mathbb{B}_i$, we define the product of $(F_i)_{i \in I}$, which we denote $\prod_i F_i$, as the pair of associations which take

$$(A_i)_i \mapsto (F(A_i))_i$$

$$(f_i)_i \mapsto (F(f_i))_i$$

for any object $(A_i)_i \in \prod_i \mathbb{A}_i$ and morphism $f_i : (A_i)_i \to (B_i)_i \in \mathrm{Hom}(\mathbb{A})$.

**Proposition 44.** *$\prod_i F_i$ as defined above is a functor from $\prod_i \mathbb{A}_i$ to $\prod_i \mathbb{B}_i$.*

*Proof.* We have that $(F(A_i))_i$ is an object of $\prod_i \mathbb{A}_i$, and $(F(f_i))_i$ is a morphism in $\prod_i \mathbb{B}_i$.

1. We have that since for each $i$, we have $F(f_i) : F(A_i) \to F(B_i)$

$$(F(f_i))_i : \left(\prod_i F_i\right)(A_i)_i \to \left(\prod_i F_i\right)(B_i)_i$$

2. Suppose that we have $(\mathrm{id}_i)_i$ is identity of $(A_i)_i$. Then we have that $F(\mathrm{id}_i)$ is identity of $F(A_i)$, so $(F(\mathrm{id}_i))_i$ is identity of $(F(A_i))_i$

3. Suppose $(f_i)_i : (A_i)_i \to (B_i)_i$, $(g_i)_i : (B_i)_i \to (C_i)_i$, $(h_i)_i : (C_i)_i \to (D_i)_i$ are morphisms. Then since for any $i$, we have $(h_i \circ g_i) \circ f_i = h_i \circ (g_i \circ f_i)$, we have

$$\begin{aligned}((h_i)_i \circ (g_i)_i) \circ (f_i)_i &= ((h_i \circ g_i)_i \circ (f_i)_i \\ &= ((h_i \circ g_i) \circ f_i)_i \\ &= (h_i \circ (g_i \circ f_i))_i \\ &= (h_i)_i \circ (g_i \circ f_i)_i \\ &= (h_i)_i \circ ((g_i)_i \circ (f_i))\end{aligned}$$

□

**Example 45.** Given the index set $I$ is given by $I = \{1, 2\}$, and functors $F : \mathbb{A} \to \mathbb{B}$, $F : \mathbb{A}' \to \mathbb{B}'$, we have that $F \times F'$ is a functor from $\mathbb{A} \times \mathbb{A}'$ to $\mathbb{B} \times \mathbb{B}'$. It takes $(A, A') \in Ob(\mathbb{A} \times \mathbb{A}')$ to

$$F \times F' : (A, A') \mapsto (F(A), F(A'))$$

and morphism $(f, f') \in \mathrm{Hom}(\mathbb{A} \times \mathbb{A}')$ to

$$F \times F' : (f, f') \mapsto (F(f), F(f'))$$

## 3.8 Bifunctors

**Definition 46.** When $F : \mathbb{A} \to \mathbb{B}$ is a functor, and the category $\mathbb{A}$ is expressed as a product of two categories, then $F$ is called a "bifunctor".

[22, page 18] mentions the important example of functors, the Hom bifunctor. We give a precise definition and prove that it is indeed a bifunctor.

**Example 47.** [22, page 18] Let $\mathbb{A}$ be a locally $\mathcal{U}$-small category, that is, all its Hom-sets are in the universal set $\mathcal{U} = Ob(\mathbf{Set})$. Let the opposite category of $\mathbb{A}$ be denoted as $\mathbb{A}^{\mathrm{op}}$. The Hom functor, which is a functor from $\mathbb{A}^{\mathrm{op}} \times \mathbb{A}$ to **Set** is defined as follows.

For $(A, B) \in Ob(\mathbb{A}^{\mathrm{op}} \times \mathbb{A})$ and for morphisms $\alpha \in \mathrm{Hom}^{\mathrm{op}}(B, A)$, $\beta \in \mathrm{Hom}(C, D)$, take

$$(A, B) \mapsto \mathrm{Hom}(A, B)$$

$$(\alpha, \beta) \mapsto (\mathrm{Hom}(B, C), \beta \circ \bullet \circ \alpha, \mathrm{Hom}(A, D))$$

where $\beta \circ \bullet \circ \alpha$ denotes the map

$$\beta \circ \bullet \circ \alpha : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, D)$$

which takes $\mu \in \mathrm{Hom}(B, C)$ to

$$\beta \circ \bullet \circ \alpha : \mu \to \beta \circ \mu \circ \alpha$$

**Proposition 48.** *The* Hom *functor as defined above, is in fact a functor.*

*Proof.* Suppose $(A, B), (A', B') \in Ob(\mathbb{A}^{\mathrm{op}} \times \mathbb{A})$, and suppose $(\alpha, \beta) : (B, C) \to (A, D)$ is a morphism.

1. We have $\mathrm{Hom}(\alpha, \beta) = (\mathrm{Hom}(B, C), \beta \circ \bullet \circ \alpha, \mathrm{Hom}(A, D))$, which is a morphism from $\mathrm{Hom}(B, C)$ to $\mathrm{Hom}(A, D)$.

2. For identity $(\mathrm{id}_A, \mathrm{id}_B) \in \mathrm{Hom}((A, B), (A, B))$, we have that $\mathrm{Hom}(\mathrm{id}_A, \mathrm{id}_B)$ corresponds to $\mathrm{id}_B \circ \bullet \circ \mathrm{id}_A : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$. Given $\alpha \in \mathrm{Hom}(A, B)$, we have

$$\mathrm{id}_B \circ \bullet \circ \mathrm{id}_A(\alpha) = \mathrm{id}_B \circ \alpha \circ \mathrm{id}_A = \alpha$$

so we see that $\mathrm{id}_B \circ \bullet \circ \mathrm{id}_A$ is the identity map and therefore corresponds to the identity element of $\mathrm{Hom}(A, B)$.

3. Suppose $(\alpha, \beta) : (B, C) \to (A, D)$ and $(\mu, \eta) : (A, D) \to (X, E)$ are morphisms in $(\mathbb{A}^{\mathrm{op}} \times \mathbb{A})$. We have that

$$\mathrm{Hom}((\alpha, \beta) \circ_{\mathbb{A}^{\mathrm{op}} \times \mathbb{A}} (\mu, \eta)) = \mathrm{Hom}((\mu \circ_{\mathrm{op}} \alpha, \eta \circ \beta) = \mathrm{Hom}((\alpha \circ \mu, \eta \circ \beta)$$

$$= \eta \circ \beta \circ \bullet \circ \alpha \circ \mu$$

This takes

$$f \mapsto \eta \circ \beta \circ f \circ \alpha \circ \mu$$

On the other hand, we have

$$\mathrm{Hom}(\alpha, \beta) \circ \mathrm{Hom}(\mu, \eta) = (\beta \circ \bullet \circ \alpha) \circ_{\mathbf{Set}} (\mu \circ \bullet \circ \eta)$$

This takes

$$f \xmapsto{(\mu \circ \bullet \circ \eta)} (\mu \circ f \circ \eta) \xmapsto{(\beta \circ \bullet \circ \alpha)} \eta \circ (\beta \circ f \circ \alpha) \circ \mu$$

So The two maps coincide.

□

### 3.9 Bifunctor Fixed With an Object

We follow the notation in [22] when defining bifunctors with one argument fixed with an object.

**Definition 49.** For bifunctor $F : I \times J \to \mathbb{A}$ and element $j \in J$ we shall denote $F(\bullet, j) : I \to \mathbb{A}$ as the functor which maps

$$F(\bullet, j)(i) \mapsto F(i, j)$$

$$F(\bullet, j)(f) \mapsto F(f, \mathrm{id}_j)$$

We denote in a similar manner $F(i, \bullet)$ the functor which does the same thing with the other argument.

**Proposition 50.** *If $F : I_1 \times I_2 \to \mathbb{A}$ is a bifunctor, then for $j \in I_2$ and $i \in I_1$, the associations $F(\bullet, j)$ and $F(i, \bullet)$ are functors.*

*Proof.* We have that for object $i \in Ob(I)$, the functor takes it to $F(i, j) \in \mathrm{Ob}(\mathbb{A})$. For morphism $f : i \to i'$, we have that $F(f, \mathrm{id}_j) \in \mathrm{Hom}(\mathbb{A})$.

1. If $f : i \to i'$ in $I$, then $F(\bullet, j)(f) = F(f, \mathrm{id}_j)$. We have that $F$ is a functor so it takes $(f, \mathrm{id}_j) : (i, j) \to (i', j)$ to $F(f, \mathrm{id}_j) : F(i, j) \to F(i', j)$.

2. For identity $\mathrm{id}_i : i \to i$ in $I$, we have that $F(\bullet, j)(\mathrm{id}_i) = F(\mathrm{id}_i, \mathrm{id}_j)$, which we already showed to be the identity of $(i, j) = F(\bullet, j)(i)$.

3. Given any two composible morphisms $f : i \to i'$, $g : i' \to i''$, we have

$$F(\bullet, j)(g \circ f) = F(g \circ f, \mathrm{id}_j) = F(g \circ f, \mathrm{id}_j \circ \mathrm{id}_j) = F((g, \mathrm{id}_j) \circ (f, \mathrm{id}_j))$$

$$= F(g, \mathrm{id}_j) \circ F(f, \mathrm{id}_j)) = F(\bullet, \mathrm{id}_j)(g) \circ F(\bullet, \mathrm{id}_j)(f)$$

We can then define new functor $G$ by reversing the position of $I_1 \times I_2$ to $I_2 \times I_1$ to obtain $G : I_2 \times I_1 \to K$ (because the order of the index does not matter; see definition 30) so that $F(i, \bullet) = G(\bullet, i)$. Then as we have proven, $G(\bullet, i)$ is a functor, so $F(i, \bullet)$ is a functor. □

**Example 51.** Given category $\mathbb{A}$ and object $A \in Ob(\mathbb{A})$, we have that $\mathrm{Hom}(A, \bullet)$ and $\mathrm{Hom}(\bullet, A)$ are functors. The functor $\mathrm{Hom}(A, \bullet)$ takes object $X \in Ob(\mathbb{A})$ to the set of all morphisms from $A$ to $X$, and given morphism $\beta : X \to Y$, we have that $\mathrm{Hom}(A, \bullet)$ takes it to the map

$$\beta \circ \bullet : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$$

$$\beta \circ \bullet : f \mapsto \beta \circ f$$

### 3.10 Argument-wise Composition of Functors with Bifunctors

Suppose $I, J, I', J'$, and $\mathbb{A}$ are categories. Given covariant bifunctor $F : I \times J \to \mathbb{A}$, and covariant functors $G : I' \to I$, $H : J' \to J$, define the bifunctor

$$F(G\bullet, H\bullet) : I' \times J' \longrightarrow \mathbb{A}$$

by mapping $(X, Y) \in Ob(I) \times Ob(J)$ to $F(G'(X), H(G'(Y))) \in Ob(\mathbb{A})$, and mapping morphisms $(\alpha, \beta) : (X, Y) \to (Z, W)$ in $I' \times J'$ to $F(G(\alpha), H(\beta))$. Then $F(G\bullet, H\bullet)$ respects identity and composition and hence is a functor as stated in the following proposition.

**Proposition 52.** *$F(G\bullet, H\bullet)$ as described above is a functor.*

*Proof.* We observe that $F(G\bullet, H\bullet)$ is simply the composition of the functors $F$ and $G \times H$ (see definition 41 and example 45). □

**Example 53.** See proposition 66, where we define $\boxtimes \circ (\boxtimes \times \mathrm{id}_{\mathbb{A}}) = \boxtimes(\boxtimes\bullet, \mathrm{id}_{\mathbb{A}}\bullet)$.

### 3.11 Natural Tranformations (Morphisms of Functors)

Let $\mathbb{A}$ and $\mathbb{B}$ denote two categories. We shall define what are commonly known as "natural transformations". These are defined as follows.

**Definition 54.** [22, page 18, 19] For functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{A} \to \mathbb{B}$, we shall say that a map $\theta : Ob(\mathbb{A}) \to \mathrm{Hom}(\mathbb{B})$ is a "natural transformation from $F$ to $G$" iff:

1. $\theta(X) : F(X) \to G(X)$ for all objects $X \in Ob(\mathbb{A})$.

2. For all $\alpha : X \to Y$ in Hom($\mathbb{A}$), the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\theta(X)"] \arrow[d, "F(\alpha)"] & G(X) \arrow[d, "G(\alpha)"] \\ F(Y) \arrow[r, "\theta(Y)"'] & G(Y) \end{tikzcd}$$

commutes.

A natural transformation $\theta$ is called an "natural isomorphism" iff $\theta(X)$ is an isomorphism for all $X$. It is more common to denote $\theta(X)$ as $\theta_X$, and we will use both notations. We will write $\theta : F \Rightarrow G$ to symbolically denote that $\theta$ is a natural transformation from $F$ to $G$.

**Example 55.** We give an example of a module isomorphism in proposition 85, which we prove to be natural in proposition 87.

**Definition 56.** The identity transformation of functor $F$ is one which takes which takes $X \in Ob(\mathbb{A})$ to the identity morphism $\mathrm{id}_F$.

We have that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\mathrm{id}_{F(X)}"] \arrow[d, "F(\alpha)"] & F(X) \arrow[d, "F(\alpha)"] \\ F(Y) \arrow[r, "\mathrm{id}_{F(Y)}"'] & F(Y) \end{tikzcd}$$

commutes for all $X, Y \in Ob(\mathbb{A})$ and morphisms $\alpha \in$ Hom, so it is indeed a natural transformation.

**Definition 57.** Given functors $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{A} \to \mathbb{B}$, and $H : \mathbb{A} \to \mathbb{B}$, and natural transformations $\theta : F \Rightarrow G$, $\eta : G \Rightarrow H$, we will denote $\eta \circ \theta$ as the map which takes

$$\eta \circ \theta : Ob(\mathbb{A}) \longrightarrow \mathrm{Hom}(\mathbb{B})$$

$$\eta \circ \theta : X \longmapsto \theta(X) \circ \eta(X)$$

Indeed, we note that since $\eta(X) : F(X) \to G(X)$ and $\theta(X) : G(X) \to H(X)$ in category $\mathbb{B}$, we have that the composition $\theta(X) \circ \eta(X)$ is defined. We see that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\theta(X)"] \arrow[d, "F(\alpha)"] & G(X) \arrow[r, "\eta(X)"] \arrow[d, "G(\alpha)"] & H(X) \arrow[d, "H(\alpha)"] \\ F(Y) \arrow[r, "\theta(Y)"'] & G(Y) \arrow[r, "\eta(Y)"'] & H(Y) \end{tikzcd}$$

commutes, so $\eta \circ \theta$ is a natural transformation from $F$ to $H$.

**Example 58.** The left and right unit constraints (see definition 70) give an example of natural isomorphisms. An explicit example of left and right unit constraints is given in 85; their naturality are proven in 87.

We mention here that given categories $\mathbb{A}$ and $\mathbb{B}$, that if we take set of all functors from $\mathbb{A}$ to $\mathbb{B}$ as objects and natural transformations as morphisms between them, this obtains a category, which can be checked by confirming necessary commutativity conditions. We do not give a proof as this is not used in defining tensor categories.

## 3.12 Subcategories

Drawing from group theory for example, the subgroup of a group is a subset of the group along with a law of composition that allows it to be viewed as a group in its own right. The law of composition must coincide with the law of composition on the original group. We apply this notion to categories.

**Definition 59.** Given categories $\mathbb{A}$, we shall say that the pair $(Ob(S), \mathrm{Hom}_S)$ is a "subcategory" of $\mathbb{A}$ iff:

1. $Ob(S) \subset Ob(\mathbb{A})$.
2. $\forall X, Y \in Ob(S) : \mathrm{Hom}_S(X, Y) \subset \mathrm{Hom}_{\mathbb{A}}(X, Y)$.
3. If $f \in \mathrm{Hom}_S(X, Y)$ and $g \in \mathrm{Hom}_S(Y, Z)$, then $g \circ_{\mathbb{A}} f \in \mathrm{Hom}_S(X, Z)$.
4. If $X \in Ob(S)$ and $\mathrm{id}_X \in \mathrm{Hom}_{\mathbb{A}}(X, X)$ is the identity morphism, then $\mathrm{id}_X \in \mathrm{Hom}_S(X, X)$.

**Example 60.** Any category is a subcategory of itself.

**Example 61.** The empty category is a subcategory of any category.

**Proposition 62.** *If $\mathbb{A}$ is a category, and $(Ob(S), \mathrm{Hom}_S)$ is a subcategory of $\mathbb{A}$, then if we define $g \circ_S f := g \circ_{\mathbb{A}} f$, and $\mathrm{Hom}(S)$ as the union of all $\{\mathrm{Hom}(A, B)\}_{A,B \in Ob(S)}$, then $(Ob(S), \mathrm{Hom}(S), \mathrm{Hom}_S, \circ_S)$ is a category.*

*Proof.* We verify the conditions.

1. We have that by condition 2 of definition 59, the Hom-sets are pairwise disjoint, and together with the definition that $\mathrm{Hom}(S)$ as the union of all $\{\mathrm{Hom}(A, B)\}_{A,B \in Ob(S)}$, we have a partition.
2. Suppose $A, B, C \in Ob(S)$.
   (a) By definition, $\circ_S$ composes from $\mathrm{Hom}_S(A, B) \times \mathrm{Hom}_S(B, C)$ and gives an element in $\mathrm{Hom}_S(A, C)$.
   (b) The identity morphism of $A \in Ob(S)$ is the identity morphism of $A$ as viewed as an object of $\mathbb{A}$.
   (c) Since composition in $S$ is defined as the same as the composition in $\mathbb{A}$, we immediately have associativity.

□

## 3.13 Functor restricted on a Subcategory

**Proposition 63.** *Given any functor $F : \mathbb{A} \to \mathbb{B}$, if $S$ is a subcategory of $\mathbb{A}$, then the pair of associations, denoted as $F \mid_S$ which associates*

$$A \mapsto F(A)$$

$$f \mapsto F(f)$$

*for any $A \in Ob(\mathbb{A})$ and $f \in \mathrm{Hom}_{\mathbb{A}}$ is a functor from $S$ to $\mathbb{B}$.*

*Proof.* Because $F$ is a functor, we have that if $f : A \to B$, then $F(f) : F(A) \to F(B)$. So $F\mid_S (f) : F\mid_S (A) \to F\mid_S (B)$. Further,

$$F\mid_S (g \circ_S f) = F(g \circ_{\mathbb{A}} f) = F(g) \circ_{\mathbb{B}} F(f)$$

and

$$F\mid_S (\mathrm{id}_A) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)} = \mathrm{id}_{F\mid_S(A)}$$

so $F\mid_S$ is a functor. □

**Proposition 64.** *Given functor $F : \mathbb{A} \to \mathbb{B}$, if $S$ is a subcategory of $\mathbb{B}$, and for all objects $A \in Ob(\mathbb{A})$, and morphisms $f \in \mathrm{Hom}(\mathbb{A})$ we have that $F(A) \in Ob(S)$ and $F(f) \in \mathrm{Hom}(S)$, then $F$ is a functor from $\mathbb{A}$ to $S$.*

*Proof.* Immediate from the definition of a functor. □

## 3.14 Tensor Categories

Tensor categories are also known as monoidal categories. A definition of the tensor category may be found in, for example, Kassel [12, page 281 onward]. We restate them below.

The tensor product is often denoted $\otimes$. We shall instead use $\boxtimes$ in our discussion of the tensor category to avoid repeating this notation.

**Definition 65.** [12, page 281] Given category $\mathbb{A}$, and a functor $\boxtimes : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, we will say that the pair $(\mathbb{A}, \boxtimes)$ is a "pre-monoidal category" (or a "pre-tensor category"). We shall say that $\boxtimes$ is the "premonoidal operation on $\mathbb{A}$".

Notationally, we shall write $A \boxtimes B$ to denote $\boxtimes(A, B)$, and write $f \boxtimes g$ to denote the morphism $\boxtimes(f, g)$.

We recall definition 43 of the product of two functors. Take $\mathrm{id}_{\mathbb{A}}$ as the identity functor (see example 37). Then $\boxtimes \times \mathrm{id}_{\mathbb{A}} : (\mathbb{A} \times \mathbb{A}) \times \mathbb{A} \to \mathbb{A} \times \mathbb{A}$. Noting that $\boxtimes : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, we can therefore compose $\boxtimes$ with $\boxtimes \times \mathrm{id}_{\mathbb{A}}$. Denote $\boxtimes(\boxtimes \times \mathrm{id}) := \boxtimes \circ (\boxtimes \times \mathrm{id}_{\mathbb{A}})$. This is a functor which associates

$$\boxtimes(\boxtimes \times \mathrm{id})(A, B, C) \mapsto (A \boxtimes B) \boxtimes C$$

$$\boxtimes(\boxtimes \times \mathrm{id})(f, g, h) \mapsto (f \boxtimes g) \boxtimes h$$

for objects $A, B, C$, and for morphisms $f, g, h \in \mathrm{Hom}(\mathbb{A})$. Now we had proved that the product and compositions of functors is a functor, but we give a didactic proof from first principles to elucidate.

**Proposition 66.** $\boxtimes(\boxtimes \times id)$ *is a functor.*

*Proof.* We have that it maps objects from $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$ to $\mathbb{A}$, and morphisms in $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$ to morphisms in $\mathbb{A}$.

1. If $(f, g, h) : (A, B, C) \to (A, B', C')$ in $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$, then we have that by assumption that $\boxtimes$ is a premonoidal operation, $\boxtimes(f, g) : A \boxtimes B \to A' \boxtimes B'$ in $\mathbb{A}$. So we have that $(f \boxtimes g) \boxtimes h : (A \boxtimes B) \boxtimes C \to (A' \boxtimes B') \boxtimes C$ in $\mathbb{A}$. That is, we have $\boxtimes(\boxtimes \times \mathrm{id})(f, g, h) : \boxtimes(\boxtimes \times \mathrm{id})(A, B, C) \to \boxtimes(\boxtimes \times \mathrm{id})(A', B', C')$

2. For an object $(A, B, C) \in Ob(\mathbb{A} \times \mathbb{A} \times \mathbb{A})$, we have that its identity is $(\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C)$. We have $\mathrm{id}_A \boxtimes \mathrm{id}_B$ is identity of $A \boxtimes B$. When we have that

$$\mathrm{id}_{A \boxtimes B} : A \boxtimes B \to A \boxtimes B$$

$$\mathrm{id}_C : C \to C$$

is identity of the objects $A \boxtimes B$ and $C$, We have that

$$(\mathrm{id}_{A \boxtimes B}, \mathrm{id}_C)$$

is identity of $(A \boxtimes B, C)$. Therefore

$$\boxtimes(\mathrm{id}_{A \boxtimes B}, \mathrm{id}_C) = \mathrm{id}_{A \boxtimes B} \boxtimes \mathrm{id}_C = (\mathrm{id}_A \boxtimes \mathrm{id}_B) \boxtimes \mathrm{id}_C = \boxtimes(\boxtimes \times \mathrm{id})(\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C)$$

is the identity of

$$\boxtimes((A \boxtimes B), C) = \boxtimes(\boxtimes \times \mathrm{id})(A, B, C)$$

3. Given any two composible morphisms $(f, g, h) : (A, B, C) \to (A', B', C')$ and $(f', g', h') : (A', B', C') \to (A'', B'', C'')$, we have that

$$\begin{aligned}
\boxtimes(\boxtimes \times \mathrm{id})((f', g', h') \circ (f, g, h)) &= \boxtimes(\boxtimes \times \mathrm{id})(f' \circ f, g' \circ g, h' \circ h) \\
&= (f' \circ f \boxtimes g' \circ g) \boxtimes h' \circ h \\
&= (\boxtimes(f' \circ f, g' \circ g)) \boxtimes h' \circ h \\
&= (\boxtimes((f', g') \circ (f, g))) \boxtimes h' \circ h \\
&= (\boxtimes(f', g') \circ \boxtimes(f, g)) \boxtimes h' \circ h \\
&= ((f' \boxtimes g') \circ (f \boxtimes g)) \boxtimes h' \circ h \\
&= \boxtimes((f' \boxtimes g') \circ (f \boxtimes g), h' \circ h) \\
&= \boxtimes((f' \boxtimes g'), h') \circ \boxtimes((f \boxtimes g), h) \\
&= (f' \boxtimes g') \boxtimes h' \circ (f \boxtimes g) \boxtimes h \\
&= \boxtimes(\boxtimes \times \mathrm{id})(f', g', h') \circ \boxtimes(\boxtimes \times \mathrm{id})(f, g, h)
\end{aligned}$$

□

On the other hand, define the functor $\boxtimes(\mathrm{id} \times \boxtimes) : \mathbb{A} \times \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ as the functor

$$\boxtimes \circ (\mathrm{id}_{\mathbb{A}} \times \boxtimes)$$

Which associates

$$\boxtimes(\mathrm{id} \times \boxtimes)(A, B, C) \mapsto A \boxtimes (B \boxtimes C)$$

$$\boxtimes(\mathrm{id} \times \boxtimes)(f, g, h) \mapsto f \boxtimes (g \boxtimes h)$$

for objects $A, B, C$, and for morphisms $f, g, h \in \mathrm{Hom}(\mathbb{A})$.

**Proposition 67.** *$\boxtimes(id \times \boxtimes)$ is a functor.*

*Proof.* The product and compositions of functors is a functor. Alternatively, we have a proof essentially identical that of proposition 66. □

In fact, we can define many functors from an $n$-product of categories

$$\mu : \mathbb{A} \times \mathbb{A} \times \cdots \times \mathbb{A} \to \mathbb{A}$$

which associates the product of objects and morphisms to the successive application of the monoidal product with respect to some configuration of brackets.

**Definition 68.** For pre-monoidal category $(\mathbb{A}, \boxtimes)$, we say that $\alpha$ is an "associativity constraint" iff it is a natural isomorphism from $\boxtimes(\boxtimes \times \mathrm{id})$ to $\boxtimes(\mathrm{id} \times \boxtimes)$. That is to say, $\alpha$ is a map from $Ob(\mathbb{A}) \times Ob(\mathbb{A}) \times Ob(\mathbb{A})$ to $\mathrm{Hom}(\mathbb{A})$ such that for all objects $A, B, C, A', B', C' \in Ob(\mathbb{B})$, and morphisms $f : A \to A', g : B \to B', h : C \to C'$, the diagram

$$\begin{tikzcd}
(A \boxtimes B) \boxtimes C \arrow[r, "\alpha(A,B,C)"] \arrow[d, "(f \boxtimes g) \boxtimes h"] & A \boxtimes (B \boxtimes C) \arrow[d, "f \boxtimes (g \boxtimes h)"] \\
(A' \boxtimes B') \boxtimes C' \arrow[r, "\alpha(A',B',C')"'] & A' \boxtimes (B' \boxtimes C')
\end{tikzcd}$$

commutes.

Although under the associativity constraint, we can view the monoidal product as two objects which are associative, we would like further constraints. Recall that in defining algebraic objects such as groups, monoids, rings, and fields and so on, when we impose associativity of an operation, we require as an axiom, the equality

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

which allows us to completely disregard brackets in multiplication, and for example, allows us to write

$$((a \cdot b) \cdot c) \cdot d = a \cdot (b \cdot (c \cdot d))$$

by repeatedly applying the associativity condition. However, what we have done in giving the associativity constraint is merely give an isomorphism (which is also natural i.e. the analogous diagram to that mentioned in definition 68 commutes), not an equality. So in fact, how we obtain $a \cdot (b \cdot (c \cdot d))$ from $((a \cdot b) \cdot c) \cdot d$ might depend on the way in which moved the brackets around. The following condition imposes that this does not matter.

**Definition 69.** [12, page 282] Given pre-monoidal Category $(\mathbb{A}, \boxtimes)$ with an associativity constraint $\alpha$, we say that it satisfies the pentagon axiom iff for all objects $A, B, C, D \in Ob(\mathbb{A})$, the following diagram of objects in $\mathbb{A}$ and morphisms in $\mathbb{A}$

$$\begin{tikzcd}
((A \boxtimes B) \boxtimes C) \boxtimes D \arrow[rr, "\alpha(A,B,C) \boxtimes \mathrm{id}_D"'] \arrow[d, "\alpha(A \boxtimes B, C, D)"'] & & (A \boxtimes (B \boxtimes C)) \boxtimes D \arrow[d, "\alpha(A, B \boxtimes C, D)"] \\
(A \boxtimes B) \boxtimes (C \boxtimes D) \arrow[dr, "\alpha(A,B,C \boxtimes D)"'] & & A \boxtimes ((B \boxtimes C) \boxtimes D) \arrow[dl, "\mathrm{id}_A \boxtimes \alpha(B,C,D)"] \\
& A \boxtimes (B \boxtimes (C \boxtimes D)) &
\end{tikzcd}$$

commutes.

In the above diagram, the compositions of the maps are in fact defined. We verify two of these, as the rest are similar to verify.

$\alpha(A, B, C)$ is a morphism from $(A \boxtimes B) \boxtimes C$ to $A \boxtimes (B \boxtimes C)$ in $\mathbb{A}$. Therefore applying the tensor product with $\mathrm{id}_D : D \to D$ on the right, we get

$$\alpha(A, B, C) \boxtimes \mathrm{id}_D : ((A \boxtimes B) \boxtimes C) \boxtimes D \to (A \boxtimes (B \boxtimes C)) \boxtimes D$$

Since $(B \boxtimes C)$ is an object of $\mathbb{A}$, and therefore $\alpha$ maps $A, B \boxtimes C, D$ to some morphism

$$\alpha(A, B \boxtimes C, D) : (A \boxtimes (B \boxtimes C)) \boxtimes D \to A \boxtimes ((B \boxtimes C) \boxtimes D)$$

An associativity contraint that satisfies the pentagon axiom in fact is enough to allow us to take any number of products of monoidal objects and whichever path we take from one configuration to the other gives the same natural isomorphism in the end. This non-trivial statement is called the Mac Lane coherence theorem, and is briefly mentioned in section 8.

**Definition 70.** [12, page 282] For pre-monoidal Category $(\mathbb{A}, \boxtimes)$, and for object $I \in Ob(\mathbb{A})$, any natural isomorphism $l : I \boxtimes \bullet \Rightarrow \mathrm{id}$ is called a "left unit constraint with respect to $I$". Similarly, any natural isomorphism $r : \bullet \boxtimes I \Rightarrow \mathrm{id}$ is called a "right unit constraint with respect to $I$".

We recall here that from definition 49 that the functor $I \boxtimes \bullet$ takes objects $A \in \mathbb{A}$ to $I \boxtimes A$, and takes morphisms $f : A \to B$ to $\mathrm{id}_I \boxtimes f : I \boxtimes A \to I \boxtimes B$, and the functor $\bullet \otimes I$ takes objects $A \in \mathbb{A}$ to $A \boxtimes I$, and takes morphisms $f : A \to B$ to $f \boxtimes \mathrm{id}_I : A \boxtimes I \to B \boxtimes I$.

Therefore, to say that $l$ and $r$ as defined above are natural transformations is to say for all $A, B \in Ob(\mathbb{A})$, and $f : A \to B$ in $\mathrm{Hom}(\mathbb{A})$, we have that $l(A), l(B), r(A), r(B)$ are isomorphisms and the two diagrams

$$\begin{tikzcd}
I \boxtimes A \arrow[r, "l(A)"'] \arrow[d, "\mathrm{id}_I \boxtimes f"'] & A \arrow[d, "f"] \\
I \boxtimes B \arrow[r, "l(B)"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \boxtimes I \arrow[r, "r(A)"'] \arrow[d, "f \boxtimes \mathrm{id}_I"'] & A \arrow[d, "f"] \\
B \boxtimes I \arrow[r, "r(B)"'] & B
\end{tikzcd}$$

commute.

Given a Pre-Monoidal Category $(\mathbb{A}, \boxtimes)$, associativity constraint $\alpha$, and object $I \in Ob(\mathbb{A})$, along with left and right unit constraints, $l, r$, we shall state that "the triangle axiom is satisfied" iff the diagram

$$\begin{tikzcd}
(A \boxtimes I) \boxtimes B \arrow[rr, "\alpha(A,I,B)"] \arrow[dr, "r_A \boxtimes \mathrm{id}_B"'] & & A \boxtimes (I \boxtimes B) \arrow[dl, "\mathrm{id}_A \boxtimes l_B"] \\
& A \boxtimes B &
\end{tikzcd}$$

commutes for all $A, B \in Ob(\mathbb{A})$.

With these definitions in mind, we now give a definition of the tensor cateogry of $\mathbb{A}$.

**Definition 71.** [12, page 282] We say that an ordered tuple $(\mathbb{A}, \boxtimes, I, \alpha, l, r)$, is a "tensor category" iff

1. $(\mathbb{A}, \boxtimes)$ is a pre-monoidal category.
2. $I$ is an object of $\mathbb{A}$.
3. $\alpha$ is an associativity constraint on $(\mathbb{A}, \boxtimes)$.
4. The pentagon axiom is satisfied.
5. $l$ is a left unit constraint.
6. $r$ is a right unit constraint.
7. The triangle axiom is satisfied.

**Example 72.** Let **Set** denote the category of sets, and $\times$ denote the operation which takes the cartesian product of two sets. Given functions $f : A \to B, g : A' \to B'$, denote $f \otimes g$ as the map which takes $(a, b) \mapsto (f(a), g(b))$. Affix some singleton set as $I = \{x\}$. Given sets $A, B, C$, take $\alpha_{A,B,C}$ as the bijection from $(A \times B) \times C$ to $A \times (B \times C)$. Denote $l$ as the map which takes $I \times A \to A$ by $l(x, a) = a$ for all $a \in A$, and $r$ as the map which takes $A \times I \to A$ by $r(a, x) = a$ for all $a \in A$. Then $(\mathbf{Set}, \times, I, \alpha, l, r)$ is a tensor category.

It will be shown that for commutative ring $R$, the tensor product of $R$-modules gives a tensor category in proposition 87. It will also be shown that the tensor product of tangles can be defined, and that it gives a tensor category in proposition 175.

### 3.15 Submonoidal Categories

**Definition 73.** Given tensor category $(\mathbb{A}, \boxtimes, I, \alpha, l, r)$ is a tensor category, and subcategory $\mathbb{B}$ of $\mathbb{A}$, define:

1. $\boxtimes_{\mathbb{B}}$ as the restriction of the functor of $\boxtimes$ on the objects and morphisms $\mathbb{B}$.
2. $I_{\mathbb{B}} := I$.
3. $\alpha_{\mathbb{B}}$ as the restriction of $\alpha$ on objects of $\mathbb{B}$.
4. $l_{\mathbb{B}}$ and $r_{\mathbb{B}}$ as the restriction of $l$ and $r$ on the objects and morphisms of $\mathbb{B}$.

**Proposition 74.** *If $(\mathbb{A}, \boxtimes, I, \alpha, l, r)$ is a tensor category and $\mathbb{B}$ is a subcategory of $\mathbb{A}$ such that:*

1. *For all objects $A, B \in Ob(\mathbb{B})$, we have $A \boxtimes B \in Ob(\mathbb{B})$ and for all $f, g \in \mathrm{Hom}\, \mathbb{B}$, we have $g \boxtimes f \in \mathbb{B}$.*
2. *$I \in \mathbb{B}$.*
3. *For all $A, B, C \in Ob(\mathbb{B})$, we have $\alpha(A, B, C) \in \mathrm{Hom}(\mathbb{B})$.*
4. *For all $A \in Ob(\mathbb{B})$, we have $l(A), r(A) \in \mathrm{Hom}(\mathbb{B})$.*

*then $(\mathbb{B}, \boxtimes_{\mathbb{B}}, I_{\mathbb{B}}, \alpha_{\mathbb{B}}, l_{\mathbb{B}}, r_{\mathbb{B}})$ is a tensor category.*

*Proof.* We verify the conditions one by one.

1. $(\mathbb{B}, \boxtimes_{\mathbb{B}})$ is a pre-monoidal category.

2. $I_{\mathbb{B}}$ is an object of $\mathbb{B}$.

3. $\alpha_{\mathbb{B}}$ is an associativity constraint on $(\mathbb{B}, \boxtimes)$.

4. The pentagon axiom is satisfied.

5. $l_{\mathbb{B}}$ is a left unit constraint.

6. $r_{\mathbb{B}}$ is a right unit constraint.

7. The triangle axiom is satisfied.

Proposition 63 together condition 1 obtains that $\boxtimes_{\mathbb{B}}$ is a functor from $\mathbb{B} \times \mathbb{B}$ to $\mathbb{A}$. Proposition 64 together with the assumption that the range $\boxtimes_{\mathbb{B}}$ of is also in $\mathbb{B}$, obtains that we have a functor from $\mathbb{B} \times \mathbb{B}$ to $\mathbb{B}$. So $(\mathbb{B}, \boxtimes_{\mathbb{B}})$ is a pre-monoidal category.

Condition 2 is by definition.

$\alpha_{\mathbb{B}}$ is an associativity constraint iff it is a map from $Ob(\mathbb{B}) \times Ob(\mathbb{B}) \times Ob(\mathbb{B})$ to $\mathrm{Hom}(\mathbb{B})$ such that for all objects $A, B, C, A', B', C' \in Ob(\mathbb{B})$, and morphisms $f : A \to A', g : B \to B', h : C \to C'$ in $\mathbb{B}$, the diagram

$$\begin{tikzcd}
(A \boxtimes B) \boxtimes C \arrow[r, "\alpha_{\mathbb{B}}(A,B,C)"] \arrow[d, "(f \boxtimes g) \boxtimes h"] & A \boxtimes (B \boxtimes C) \arrow[d, "f \boxtimes (g \boxtimes h)"] \\
(A' \boxtimes B') \boxtimes C' \arrow[r, "\alpha_{\mathbb{B}}(A',B',C')"'] & A' \boxtimes (B' \boxtimes C')
\end{tikzcd}$$

commutes. In fact, we see that this simply translates to a diagram in $\mathbb{A}$, so it commutes. So 3 is shown.

Again, the pentagon diagram in $\mathbb{B}$ translates to a diagram in $\mathbb{A}$, so it commutes. So 4 is satisfied.

An isomorphism in $\mathbb{A}$ is also an isomorphism in $\mathbb{B}$. Therefore $l_{\mathbb{B}}$ and $r_{\mathbb{B}}$ both take objects in $\mathbb{B}$ to isomorphisms in $\mathbb{B}$. Further, the desired diagrams commute because they commute in $\mathbb{A}$. So 5 and 6 are satisfied.

Finally, since the triangle diagram commutes in $\mathbb{A}$, it also commutes in $\mathbb{B}$. $\square$

If the conditions in proposition 74 are true, will state that $(\mathbb{B}, \boxtimes_{\mathbb{B}}, I_{\mathbb{B}}, \alpha_{\mathbb{B}}, l_{\mathbb{B}}, r_{\mathbb{B}})$ as given in definition 73 is a "submonoidal category" of $\mathbb{A}$.

# 4 Tensor Product of $R$-Modules

In this section we recall the algebraic definition of modules. We show that the set of all $\mathcal{U}$-small $R$-modules, along with their homomorphisms forms a category. Then we define the categorical notion of a tensor product of two $R$-modules via a universal property, and explicitely exhibit an initial object which we affix as the tensor product. We then proceed to show that $R$-modules, equipped with the tensor product that we have defined forms a monoidal category.

## 4.1 Rings and Modules and Algebras over a Ring

We first remind ourselves of basic definition of modules, which is here presented in compact form.

**Definition 75.** A left $R$-module is an orderered triple $(M, R, \rho)$, such that:

1. $R$ is a ring.
2. $M$ is an Abelian group.
3. $\rho : (R, \cdot) \to End_{\mathbf{Grp}}(M)$ is a monoid homomorphism; it defines a left monoid action of $(R, \cdot)$ on $M$.
4. For all $a, b \in R$ and $x, y \in M$ the action satisfies:
   (a) $a \cdot (x + y) = a \cdot x + a \cdot y$.
   (b) $(a + b) \cdot x = a \cdot x + b \cdot x$.

Similarly, a right $R$-module is an orderered triple $(M, R, \rho)$, where instead $\rho$ is a right monoid action with the approrpriate distribution properties. When $R$ is commutative, a right module is left and conversely. A map which preserves addition and action is called a "$R$-module homomorphism" or a "$R$-linear map".

We have noted that the set of all left $R$-modules forms a category, denoted $\mathbf{Mod}(R)$, although we do not give a proof.

**Definition 76.** [9, page 126] For ring $R$, right $R$-module $N$, left $R$-module $M$, and Abelian group $G$, a map $f : N \times M \to G$ is said to be "$R$-balanced" iff:

1. $f(n + n', m) = f(n, m) + f(n', m)$.
2. $f(n, m + m') = f(n, m) + f(n, m')$.
3. $f(n \cdot r, m) = f(n, r \cdot m)$.

**Definition 77.** Adapted from [22, page 9]. A ordered triple $(R, k, \varphi)$ is said to be a "$k$-algebra" iff:

1. $R$ is a ring.
2. $k$ is a commutative ring.
3. $\varphi$ is a ring homomorphism from $k$ to $Z(R)$.

where $Z(R)$ denotes the center of $R$.

For brevity, we shall say that $R$ is a $k$-algebra for short.

Given $\lambda \in k$ and $a \in R$ we shall write use the notation $\lambda \cdot a := \varphi(\lambda) \cdot a$; and $a \cdot \lambda := a \cdot \varphi(\lambda)$. For $k$-algebra $R$, $R$-module $M$, and $\lambda \in k$, $u \in M$, we shall, for brevity, denote $\lambda \cdot u := \varphi(\lambda) \cdot u$. However, when there might be confusion, we shall explicitely write $\varphi(\lambda)$. So that indeed, $M$ is also simultaneously a (two-sided) $k$-module.

We shall define the tensor product of two $R$-modules via a universal property.

## 4.2 The Tensor Product of Two Modules

We now proceed to the definition and explicit construction of the tensor product. The definition of the tensor product which I present (definition 79) follows the methods employed by [14] in defining universal objects such as products and coproducts.

**Definition 78.** [22, page 12] For $k$-algebra $(R, k, \varphi)$, right $R$-module $N$, left $R$-module $M$, $k$-module $L$, a map $f : N \times M \to L$ is said to be an $(R, k)$-bilinear map iff:

1. It is $R$-balanced.

2. $\forall \lambda \in k : f(n\varphi(\lambda), m) = \lambda f(n, m)$.

**Definition 79.** For $k$-algebra $(R, k, \varphi)$, right $R$-module $N$, and left $R$-module $M$, denote $\mathbf{Tens}(N, M)$ as the category whose objects are all $(R, k)$-bilinear maps $f : N \times M \to L$ for some $k$-module $L$, and whose morphisms from two objects $f : N \times M \to L$ to $g : N \times M \to L'$ consist of all $k$-module homomorphisms $\alpha$ making the diagram

$$\begin{tikzcd} N \times M \arrow[r, "f"] \arrow[dr, "g"'] & L \arrow[d, "\alpha"] \\ & L' \end{tikzcd}$$

commute. An initial object of $\mathbf{Tens}(N, M)$ is said to be a "tensor product" of $N$ and $M$, and it is guaranteed to exist, by the following proposition.

[22, page 13] gives an explicit definition of a tensor product of $R$-modules, which we show to satisfy the universal property described above.

**Proposition 80.** *The category* $\mathbf{Tens}(N, M)$ *has an initial object.*

*Proof.* Denote

$$Free_k(I) := \{f : I \to k \mid f(i) = 0 \text{ for all but finite } i \in I\}$$

We note that $Free_k(I)$ is a ring with the componentwise addition and multiplication of functions, and by defining $(\lambda f)(x) = \lambda(f(x))$ for $\lambda \in k, f \in Free_k(I), x \in I$, it can be regarded as a $k$-module. Consider set $I$ as a subset of $Free_k(I)$ by the embedding $i \mapsto \delta_i$, where $\delta_i$ is defined as follows

$$\delta_i(j) := \begin{cases} 1, & \text{if } i = j. \\ 0, & \text{otherwise.} \end{cases}$$

Therefore any element $i \in I$ will be denoted as $i$ itself in the set $Free_k(I)$. We turn our attention to $Free_k(N \times M)$. Since an element $f$ in that set satisfies $f(x) = 0$ for all but finite $x \in N \times M$, we have that it is expressible as a finite sum

$$x = \sum_i \lambda_i(n_i, m_i)$$

for $\lambda_i \in k, n_i \in N, m_i \in M$, where by $(n_i, m_i)$, we mean $\delta_{(n_i,m_i)}$. Now denote $T_{N,M}$ as the set containing all elements of $Free_k(N \times M)$ of the form

$$(*)\begin{cases}(n+n',m)-(n,m)-(n',m)\\(n,m+m')-(n,m)-(n,m')\\(na,m)-(n,am)\\\lambda\cdot(n,m)-(n\lambda,m)\end{cases}\tag{1}$$

Denote $\langle T_{N,M}\rangle$ as the module generated by the set of all elements in $T_{N,M}$. Explicitely,

$$\langle T_{N,M}\rangle=\left\{\sum a_it_i:\ \text{finite sum such that } a_i\in k, t_i\in T\right\}$$

For short, denote $T_{N,M}$ as $T$.

Now we shall show that $Free_k(N\times M)/\langle T\rangle$ satisfies the desired conditions.

Denote the canonical $k$-module homomorphism as $\otimes : Free_k(N\times M)\to Free_k(N\times M)/\langle T\rangle$ which associates $x\mapsto x+\langle T\rangle$. For $(n,m)\in Free_k(I)$, we shall write

$$n\otimes m:=\otimes(n,m)=(n,m)+\langle T\rangle$$

We note that an arbitrary element in $Free_k(N\times M)/\langle T\rangle$ is expressible as

$$\sum_i\lambda_i(n_i,m_i)+\langle T\rangle$$

or as

$$\sum_i\lambda_i(n_i\otimes m_i)$$

Now, it transpires that:

$$(**)\begin{cases}n+n'\otimes m=n\otimes m+n'\otimes m\\n\otimes m+m'=n\otimes m+n\otimes m'\\na\otimes m=n\otimes am\\\lambda\cdot(n\otimes m)=n\lambda\otimes m\end{cases}$$

Then the restriction of $\otimes$ on the set $N\times M$, viewed as a subset of $Free_k(N\times M)$, is the desired object in the category $\mathbf{Tens}(N,M)$; we denote the restriction as $\otimes$ itself. Denote $N\otimes M:=Free_k(I)/\langle T\rangle$. There are three things we need to verify; that $\otimes : N\times M\to N\otimes M$ is indeed an object of $\mathbf{Tens}(N,M)$, and that it is indeed initial.

First, we immediately see that by $(**)$, $\otimes$ is $(R,k)$-bilinear so it is an object of $\mathbf{Tens}(N,M)$.

Now suppose $f : N\times M\to L$ is an object of $\mathbf{Tens}(N,M)$. We find $\alpha$ making the diagram

$$\begin{tikzcd}N\times M \arrow[r, "\otimes"] \arrow[dr, "f"'] & N\otimes M \arrow[d, "\alpha"]\\ & L'\end{tikzcd}$$

commute. Define $\alpha$ as follows:

$$\alpha:\sum\lambda_i(n_i,m_i)+\langle T\rangle\mapsto\sum\lambda_if(n_i,m_i)$$

Now we show that this association is unique, i.e. it is well defined. Suppose $\sum \lambda_i(n_i, m_i) + \langle T \rangle = \sum \lambda'_i(n_i, m_i) + \langle T \rangle$. Then $\sum(\lambda_i - \lambda'_i)(n_i, m_i) \in \langle T \rangle$. Then it can be expressed as a finite sum $\sum b_i t_i$ such that $b_i \in k, t_i \in T$. Then $t_i$ is expressed in one of the forms in $(*)$. For example, if $t_i = (n + n', m) - (n, m) - (n', m)$, then $b_i t_i = b_i(n + n', m) - b_i(n, m) - b_i(n', m)$, and applying $f$ gives $b_i f(n + n', m) - b_i f(n, m) - b_i f(n', m)$; due to $(R, k)$-bilinearity, we then get $b_i f(n, m) + b_i f(n', m) - b_i f(n, m) - b_i f(n', m) = 0$. So $f(\sum b_i t_i) = \sum b_i f(t_i)) = 0$ due to $(R, k)$-bilinearity.

Since $\sum(\lambda_i - \lambda'_i)(n_i, m_i) = \sum b_i t_i$, we have

$$\sum \lambda_i(n_i, m_i) = \sum \lambda_i(n_i, m_i) + \sum b_i t_i$$

Applying $f$ on either sides gives

$$\sum \lambda_i f(n_i, m_i) = \sum \lambda_i f(n_i, m_i) + 0$$

So $\alpha$ is well defined.

Clearly, $\alpha$ makes the desired diagram commutative, we have that $\alpha \in \mathrm{Hom}(\otimes, f)$. Since any $k$-module homomorphism $\beta : L \to L'$ satisfiying the commutativity condition needs to map

$$\sum \lambda_i(n_i, m_i) + \langle T \rangle \mapsto \sum \lambda_i f(n_i, m_i)$$

we have that $\alpha = \beta$, and so $\alpha$ is unique. So indeed, $\otimes$ is initial. $\square$

The initial object in $\mathbf{Tens}(N, M)$ as explicitely constructed above in the previous proposition is denoted as $\otimes : N \times M \to N \otimes M$. The module $N \otimes M$ is called the "tensor product" of $N$ and $M$. Whenever we refer to "the tensor product of $N$ and $M$, we refer to this particular object. Whenever we say "a tensor product of $N$ and $M$", we refer to an arbitrary initial object of the category $\mathbf{Tens}(N, M)$, which is isomorphic $\otimes$ in the category. In fact, we see that such an isomorphism coincides with a $k$-module isomorphism to/from $N \otimes M$. However, we will not mention any other types of tensor products in this paper.

Instead of referring to the $(R, k)$-bilinear map $\otimes : N \times M \to N \otimes M$ as the tensor product, one often refers to the $k$-module $N \otimes M$ itself as the tensor product. We understand this to mean that "$k$-module $N \otimes M$ together with the map $\otimes$ is the tensor product".

### 4.3 Tensor Product over a Commutative Ring

I have found that to turn $R$-modules in a monoidal category we need $R$ to be commutative. We now consider the case of modules over a commutative ring $R$. It is then possible to define an $R$-module structure on the tensor product of two $R$-modules. For this, I give the following proposition.

**Proposition 81.** *For commutative ring $R$, and modules $N, M$ over $R$, and tensor product $N \otimes M$, given an element $a_i(n_i, m_i) + \langle T \rangle$ in $N \otimes M$, the association*

$$\sum a_i(n_i, m_i) + \langle T \rangle \mapsto \sum a_i(n_i r, m_i) + \langle T \rangle$$

*is well defined.*

*Proof.* We observe that every element of the form noted in equation 1, when applied by the $R$-action remains in the form within equation 1. Therefore it is invariant under $\langle T\rangle$. $\square$

When $R$ is commutative, this in fact defines a monoidal action of $(R, \cdot)$ on $N \otimes M$. For we notice that, if $1_R \in R$ is the multiplicative identity, then

$$1_R \sum a_i(n_i, m_i) + \langle T\rangle = \sum a_i(n_i 1_R, m_i) + \langle T\rangle = \sum a_i(n_i, m_i) + \langle T\rangle$$

and for $r, s \in R$, we have

$$r\left(s \sum a_i(n_i, m_i) + \langle T\rangle\right) = r\left(\sum a_i(n_i r, m_i) + \langle T\rangle\right) = \sum a_i((n_i s)r, m_i) + \langle T\rangle$$

$$= \sum a_i(n_i(rs), m_i) + \langle T\rangle = (rs) \sum a_i(n_i, m_i) + \langle T\rangle$$

Denote the $R$ action on $N \otimes M$ as $\cdot_R$, and the $k$-action on $N \otimes M$ as $\cdot_k$. Then we see that for $\lambda \in k$, we have that

$$\varphi(\lambda) \cdot_R (m \otimes n) = (m\varphi(\lambda)) \otimes n = \lambda \cdot_k (m \otimes n)$$

That is to say, $\lambda \in k$ can be multiplied to an element as either an element from $k$ or from $R$. From this, observe that we can do

$$\lambda \cdot_k (r \cdot_R (m \otimes n)) = \varphi(\lambda) \cdot_R (r \cdot_R (m \otimes n)) = (\varphi(\lambda) r) \cdot_R (m \otimes n)$$

$$r \cdot_R (\lambda \cdot_k (m \otimes n)) = r \cdot_R (\varphi(\lambda) \cdot_R (m \otimes n)) = (r\varphi(\lambda)) \cdot_R (m \otimes n)$$

$$\text{and hence, } r \cdot_R (\lambda \cdot_k (m \otimes n)) = \lambda \cdot_k (r \cdot_R (m \otimes n))$$

Hence we can write $\lambda r(m \otimes n))$ or $r\lambda(m \otimes n))$ without confusion about multiplication in $R$ or in $k$ or the brackets that we are using.

From our discussion it is possible to view the action of $\lambda \in k$ as identical to the action of $\varphi(\lambda) \in R$.

**Definition 82.** Given modules $N$ and $M$, we are allowed to consider $(N \otimes M)$ not only as a $k$-module, but also as an $R$ module by defining

$$r(n \otimes m) := (nr \otimes m)$$

### 4.4 Tensor Product of Three Modules over a Commutative Ring

Let $R$ be a commutative ring. Let $(R, k, \varphi)$ be an $k$-algebra and $L, M, N$ be $R$-modules. Let $\lambda \in k, l \in L, m \in M, n \in N$.

Note that since the action $\lambda \in k$ on $(m \otimes n)$ is the same as the action of $\varphi(\lambda) \in R$ on $(m \otimes n)$, we get $(l \otimes \lambda (m \otimes n)) = (l \otimes \varphi(\lambda) (m \otimes n)) = \lambda (l \otimes (m \otimes n))$. Therefore any element in $L \otimes (M \otimes N)$ is expressible as $\sum_i \lambda_i (l_i \otimes (m_i \otimes n_i))$ where $\lambda_i \in k, l_i \in L, m_i \in M, n_i \in N$.

Schapira [22, page 14] mentions the following fact, which we restate in precise terms as a proposition, and provide a detailed proof.

**Proposition 83.** *[22, page 14] Let $(R, k, \varphi)$ be a $k$-algebra, and $R$ be commutative, and $L, M, N$ be modules over $R$. If we consider $(L \otimes M)$ and $(M \otimes N)$ as $R$-modules, then the map $\alpha_{L,M,N} : (L \otimes M) \otimes N \to L \otimes (M \otimes N)$ which takes*

$$\sum \lambda_i (l_i \otimes m_i) \otimes n_i \longmapsto \sum \lambda_i (l_i \otimes (m_i \otimes n_i))$$

$$\textit{where } \lambda_i \in k; l_i \in L, m_i \in M, n_i \in N$$

*is a well defined $k$-module isomorphism. Further, regarding $(L \otimes M) \otimes N$ and $L \otimes (M \otimes N)$ as $R$-modules as opposed to $k$-modules, $\alpha_{L,M,N}$ is also an $R$-module isomorphism.*

*Proof.* Denote $\delta_\ell : M \otimes N \to (L \otimes M) \otimes N$ as the map which takes

$$\delta_l : \sum_i \lambda_i (m_i \otimes n_i) \longmapsto \sum_i \lambda_i ((\ell \otimes m_i) \otimes n_i)$$

for $l \in L, m_i \in M, n_i \in N$. Then $\delta_l$ is well defined, for it kills $\langle T_{M,N} \rangle$. For example, $m + m' \otimes n$, which is represented by $(m + m', n)$ becomes

$$\left(\left(\ell \otimes \left(m + m'\right)\right), n\right) = \left(\left(\ell \otimes m + l \otimes m'\right), a\right)$$

$$\sim ((\ell \otimes m), n) + \left(\left(l \otimes m'\right), n\right)$$

and so on. $\delta_l$ is clearly an $R$-module homomorphism. Now $M \otimes N$ itself is an $R$-module, so there exists universal object in $\mathbf{Tens}(L, N \otimes M)$

$$\gamma : L \times (M \otimes N) \to L \otimes (M \otimes N)$$

Consider map

$$b : L \times (M \otimes N) \to (L \otimes M) \otimes N$$

defined by associating

$$b : \left(\ell, \sum_i \lambda_i (m_i \otimes n_i)\right) \longmapsto \delta_l \left(\sum_i \lambda_i (m_i \otimes n_i)\right)$$

Then $b$ is $(R, k)$-bilinear:

Put $\mu := \sum_i \lambda_i (m_i \otimes n_i)$, and $\mu' := \sum_i \lambda'_i (m'_i \otimes n'_i)$. Then

$$b(l + l', \mu) = \delta_{l+l'}(\mu) = \sum_i \lambda_i((l + l') \otimes m_i) \otimes n_i = \sum_i \lambda_i \left(l \otimes m_i + l' \otimes m_i\right) \otimes n_i$$

$$= \sum_i \lambda_i (l \otimes m_i) \otimes n_i + \sum_i \lambda_i \left(l' \otimes m_i\right) \otimes n_i = b(l, \mu) + b(l', \mu)$$

Similarly, $b(l, \mu + \mu') = b(l, \mu) + b(l, \mu')$. For $r \in R$, we have

$$b(lr, \mu) = \delta_{lr}(\mu) = \sum_i \lambda_i ((lr \otimes m_i) \otimes n_i) = \sum_i \lambda_i (r(\ell \otimes m_i)) \otimes n_i$$

$$= \sum_i \lambda_i (\ell \otimes m_i) \otimes rn_i = b(l, \sum_i \lambda_i (m_i \otimes rn_i)) = b(l, r\mu)$$

And hence for $\lambda \in k$

$$b(l\varphi(\lambda), \mu) = \sum_i \lambda_i[(l\varphi(\lambda) \otimes m_i) \otimes n_i] = \sum_i \lambda_i[(\varphi(\lambda)(l \otimes m_i)) \otimes n_i]$$

$$= \sum_i \lambda_i(\lambda[(l \otimes m_i) \otimes n_i]) = \lambda b(l, \mu)$$

So $b$ is $(R, k)$-bilinear whence there exists unique morphism $\beth : (L \otimes M) \otimes N \to L \otimes (M \otimes N)$ where the diagram

$$\begin{tikzcd}
L \times (M \otimes N) \arrow[r, "\gamma"] \arrow[dr, "b"] & L \otimes (M \otimes N) \arrow[d, "\alpha"] \\
& (L \otimes M) \otimes N
\end{tikzcd}$$

which takes

$$\beth : \sum_i \lambda_i \left(l_i \otimes (m_i \otimes n_i)\right) \longmapsto \sum_i \lambda_i b\left(l_i, (m_i \otimes n_i)\right)$$

$$= \sum_i \lambda_i \delta_{l_i}\left(m_i \otimes n_i\right) = \sum_i \lambda_i \left(l_i \otimes m_i\right) \otimes n_i$$

Via a similar process, we also obtain a $k$-linear map $\daleth : (L \otimes M) \otimes N \to L \otimes (M \otimes N)$, which takes

$$\sum \lambda_i(l_i \otimes (m_i \otimes n_i)) \longmapsto \sum \lambda_i(l_i \otimes m_i) \otimes n_i$$

Then the compositions $\beth \circ \daleth$ and $\daleth \circ \beth$ are both identity, and therefore $\beth$ is isomorphic. So we see that it is the desired map $\alpha_{L,M,N}$.

Finally we need to show that the map preserves $R$ action. Suppose $r \in R$.

$$\alpha_{L,M,N}\left[\sum \lambda_i(l_i r \otimes (m_i \otimes n_i))\right] = \sum \lambda_i(l_i r \otimes m_i) \otimes n_i$$

$$= \sum \lambda_i(l_i \otimes m_i)r \otimes n_i$$

$$= r \sum \lambda_i(l_i \otimes m_i) \otimes n_i$$

$$r \cdot \alpha_{L,M,N}\left[\sum \lambda_i(l_i \otimes (m_i \otimes n_i))\right]$$

and the claim is shown. $\square$

**Corollary 84.** *Given a finite sequence of $R$-modules $(M_i)_{i \in I}$, and two methods of bracketing, whose results we denote as $N$ and $N'$, there exists a unique $R$-module isomorphism from $N$ to $N'$ which maps*

$$(\cdots(x_1 \otimes \cdots \otimes x_n)\cdots) \mapsto (\cdots(x_1 \otimes \cdots \otimes x_n)\cdots)$$

*where the element on the left is a sequence of $(x_1, \cdots, x_n)$ with the same bracketing of the tensor product in $N$, and the element on the right is a sequence of $(x_1, \cdots, x_n)$ with the same bracketing of the tensor product in $N'$. This is called the "canonical" module isomorphism from $N$ to $N'$.*

### 4.5 Tensoring with Ring $R$

[22] mentions that we are able to tensor an $R$-module with $R$. We give a detailed description of this here in this subsection.

A ring $R$ can be considered a left $R$ module itself by defining the monoid $(R, \cdot)$-action on $R$ as simply the multiplication on $R$. That is, given $r \in R$, and $x \in R$, the action $\rho : (R, \cdot) \to \mathrm{End}(R)$ is defined by

$$\rho_r(x) := rx$$

or when we denote action as $\cdot_R$,

$$r \cdot_R x := rx$$

Similarly, $R$ can be considered a right $R$- module by defining the action as

$$\rho_r(x) := xr$$

or when we denote action as $\cdot_R$,

$$\text{i.e. } x \cdot_R r := xr$$

Given $k$-algebra $(R, k, \varphi)$, we also notice that $R$ is a $k$-module by defining the monoid $k$-action on $R$ as:

$$\rho_\lambda(x) := \varphi(\lambda)x$$

Again, we remind ourselves that we write $\lambda x$ for $\varphi(\lambda)x$.

**Proposition 85.** *Now suppose we have $k$-algebra $(R, k, \varphi)$, where $R$ is commutative, and $R$-module $M$. Then there exists two $R$-module isomorphisms (and hence also $k$-module isomorphisms)*

$$R \otimes M \approx M \approx M \otimes R$$

*Proof.* Given $a \in R$ and $m_i \in M$, denote $l : R \otimes M \to M$ as the following map:

$$l : \sum_i \lambda_i(a_i \otimes m_i) \mapsto \sum_i \lambda_i a_i m_i$$

We show that $l$ is in fact well defined. It suffices to show that $l$ kills $\langle T \rangle$. Recall from equation 1 that in $\langle T \rangle$ we have the elements generated by the forms

$$\begin{cases} (n + n', m) - (n, m) - (n', m) \\ (n, m + m') - (n, m) - (n, m') \\ (na, m) - (n, am) \\ \lambda \cdot (n, m) - (n\lambda, m) \end{cases}$$

For $n, n' \in R$, $m, m' \in M$, and $\lambda \in k$. So applying $l$ gives

$$\begin{cases} (n + n')m - nm - n'm \\ n(m + m') - nm - nm' \\ (na)m - n(am) \\ \lambda \cdot (nm) - (n\lambda)m \end{cases}$$

and we see that all these elements are $0_M$. Therefore $l$ is indeed well defined.

Now we show that $l$ is a module homomorphism. By definition, it clearly preserves addition. Suppose $r \in R$

$$l\left(r\sum_i \lambda_i(a_i \otimes m_i)\right) = l\left(\sum_i \lambda_i(a_i r \otimes m_i)\right) = \sum_i \lambda_i a_i r m_i = r\sum_i \lambda_i a_i m_i$$
$$= r \cdot l\left(\sum_i \lambda_i(a_i \otimes m_i)\right)$$

For bijectivity, we define an inverse map $l^{-1} : M \to R \otimes M$ by putting

$$l^{-1} : m \mapsto 1_R \otimes m$$

and therefore

$$l(l^{-1}(m)) = l(1_R \otimes m) = 1_R m = m$$
$$l^{-1}\left(l\left(\sum_i \lambda_i(a_i \otimes m_i)\right)\right) = l^{-1}\left(\sum_i \lambda_i a_i m_i\right) = 1_R \otimes \sum_i \lambda_i a_i m_i$$
$$= \sum_i 1_R \otimes \lambda_i a_i m_i = \sum_i \lambda_i a_i \otimes m_i$$

So $l \circ l^{-1} = \mathrm{id}$, $l^{-1} \circ l = \mathrm{id}$, and isomorphism $R \otimes M \approx M$ is shown. Similarly, by defining the map

$$r : \sum_i \lambda_i(m_i \otimes a_i) \mapsto \sum_i \lambda_i a_i m_i$$

yields an $R$-module isomorphism $M \otimes R \approx M$. □

The isomorphisms $l$ and $r$ as defined in the above proof shall be called "canonical".

**Corollary 86.** *Given a finite sequence of $R$-modules $(M_i)_{i\in I}$, where $M_j$ is equal to $R$, then there exists unique $R$-module isomorphism*

$$x_1 \otimes \cdots x_{j-1} \otimes x_j \otimes x_{j+1} \cdots \otimes x_n \mapsto x_1 \otimes \cdots x_{j-1} \otimes x_{j+1} \cdots \otimes x_n$$

*for some configuration of brackets of $(M_i)_{i\in I}$. This isomorphism is called "canonical".*

### 4.6 The Tensor Category of $R$-Modules

From the previous discussions, I observe that $R$-modules along with the tensor product forms a tensor category. This is made precise in the next proposition.

**Proposition 87.** *Let $(R, k, \varphi)$ be a $k$-algebra, where $R$ is commutative. Denote $\mathbf{Mod}(R)$ as the category of all $R$-modules Denote $\otimes$ as the map which takes two modules $N, M$ and maps them to their tensor product $N \otimes M$. Denote $I$ as the $R$-module $R$ itself. Denote $\alpha$ as the map which takes $(L, N, M) \in Ob(R\mathbf{Mod}) \times Ob(R\mathbf{Mod}) \times Ob(R\mathbf{Mod})$ to the canonical $R$-module isomorphism $\alpha_{L,M,N} : (L \otimes M) \otimes N \to L \otimes (M \otimes N)$. Denote $l(M) : R \otimes M \to M$ and $r(M) : M \otimes R \to M$ as the canonical $R$-module isomorphisms.*

*Then $(R\mathbf{Mod}, \otimes, I, \alpha, l, r)$, is a tensor category.*

*Proof.* Recall definition 71 that we need to verify the following:
$(\mathbf{Mod}(R), \otimes, R, \alpha, l, r)$, is a tensor category iff

1. $(\mathbf{Mod}(R), \otimes)$ is a pre-monoidal category.
2. $R$ is an object of $\mathbf{Mod}(R)$.
3. $\alpha$ is an associativity constraint on $(\mathbf{Mod}(R), \otimes)$.
4. The pentagon axiom is satisfied.
5. $l$ is a left unit constraint.
6. $r$ is a right unit constraint.
7. The triangle axiom is satisfied.

We see that $\otimes$ takes objects $N, M \in Ob(\mathbf{Mod}(R))$ to $N \otimes M \in Ob(\mathbf{Mod}(R))$. Given $(N, M), (N', M') \in Ob(\mathbf{Mod}(R)) \times Ob(\mathbf{Mod}(R))$, and morphism

$$(f, g) : (N, M) \to (N', M')$$

in $\mathrm{Hom}(\mathbf{Mod}(R) \times \mathbf{Mod}(R))$, we have that we can define the map $f \times g : N \times M \to N' \times M'$ which takes

$$f \times g : (n, m) \mapsto (f(n), g(m))$$

and we see that this gives an $R$-module homomorphism. Now note that $\otimes : N' \times M' \to N' \otimes M'$ is an $(R, k)$-bilinear map. We show that $\otimes \circ (f \times g) : N \times M \to N' \otimes M'$ is also an $(A, k)$-bilinear map. It is $R$- balanced:

1. $\otimes \circ (f \times g)(n + n', m) = \otimes(f(n + n'), g(m)) = \otimes(f(n) + f(n'), g(m)) = \otimes(f(n), g(m)) + \otimes(f(n'), g(m)) = \otimes \circ (f \times g)(n, m) + \otimes \circ (f \times g)(n', m)$.
2. $\otimes \circ (f \times g)(n, m + m') = \otimes(f(n), g(m + m')) = \otimes(f(n), g(m) + g(m')) = \otimes(f(n), g(m)) + \otimes(f(n), g(m')) = \otimes \circ (f \times g)(n, m) + \otimes \circ (f \times g)(n, m')$.
3. $\otimes \circ (f \times g)(n \cdot r, m) = \otimes(f(nr), g(m)) = \otimes(f(n)r, g(m)) = \otimes(f(n), rg(m)) = \otimes(f(n), g(rm)) = \otimes \circ (f \times g)(n, r \cdot m)$.

We also have that for $\lambda \in k$ :

$$\otimes \circ (f \times g)(n\varphi(\lambda), m) = \otimes(f(n\varphi(\lambda)), g(m)) = \otimes(f(n)\varphi(\lambda), g(m))$$

$$= \lambda \cdot \otimes(f(n), g(m)) = \lambda(\otimes \circ (f \times g)(n, m))$$

and thus we have that the map is bilinear. So by the universal property of the tensor product, there exists unique $\alpha$ such that the diagram

$$\begin{tikzcd} N \times M \arrow[r, "\otimes"] \arrow[d, "f \times g"] & N \otimes M \arrow[d, "\alpha"] \\ N' \times M' \arrow[r, "\otimes"] & N' \otimes M' \end{tikzcd}$$

commutes, and it is in fact, defined by

$$\alpha\left(\sum_i \lambda_i\,(n_i\otimes m_i)\right) = \sum_i \lambda_i(\otimes\circ(f\times g))\,(n_i,m_i)$$
$$= \sum_i \lambda_i\,(f(n_i)\otimes g(m_i))$$

We shall denote this function as $f\otimes g$. Let $\otimes$ take morphim $(f,g)\in \mathrm{Hom}(\mathbf{Mod}(R)\times\mathbf{Mod}(R))$ to $f\otimes g\in\mathrm{Hom}(\mathbf{Mod}(R))$.

Now noting that for identity $(\mathrm{id}_M,\mathrm{id}_N)\in\mathrm{Hom}(\mathbf{Mod}(R)\times\mathbf{Mod}(R))$, we have that

$$\mathrm{id}\otimes\mathrm{id}\left(\sum_i \lambda_i\,(n_i\otimes m_i)\right) = \sum_i \lambda_i\,(\mathrm{id}(n_i)\otimes\mathrm{id}(m_i))$$

and therefore $\mathrm{id}\otimes\mathrm{id}$ is the identity morphism on $N\otimes M$. Further, for $(f,g):(N,M)\to(N',M')$ and $(f',g'):(N',M')\to(N'',M'')$:

$$f'\circ f\otimes g'\circ g\left(\sum_i \lambda_i\,(n_i\otimes m_i)\right) = \sum_i \lambda_i\left(f'fn_i\otimes g'gm_i\right)$$
$$= f'\otimes g'\left(\sum_i \lambda_i\,(fn_i\otimes gm_i)\right) = (f'\otimes g')(f\otimes g)\left(\sum_i \lambda_i\,(n_i\otimes m_i)\right)$$

and therefore we have the equality of morphisms $f'\circ f\otimes g'\circ g = (f'\otimes g')\circ(f\otimes g)$.

And therefore $\otimes$ is a functor. This shows 1.

We see that 2. is true.

We remind ourselves that to say that $\alpha$ is an associativity constraint is to say that for all objects $L,M,N,L',M',N'\in Ob(\mathbf{Mod}(R))$, and morphisms $f:L\to L', g:M\to M', h:N\to N'$, the diagram of morphisms in $\mathbf{Mod}(R)$:

$$\begin{tikzcd}
(L\otimes M)\otimes N \arrow[r, "\alpha(L,M,N)"] \arrow[d, "(f\otimes g)\otimes h"] & L\otimes(M\otimes N) \arrow[d, "f\otimes(g\otimes h)"] \\
(L'\otimes M')\otimes N' \arrow[r, "\alpha(L',M',N')"'] & L'\otimes(M'\otimes N')
\end{tikzcd}$$

commutes. We have that for any element $\sum_i \lambda_i\,((l_i\otimes m_i)\otimes n_i)$ in $(L\otimes M)\otimes N$, the diagram

$$\begin{tikzcd}
\sum_i \lambda_i\,((l_i\otimes m_i)\otimes n_i) \arrow[r, mapsto, "\alpha(A,B,C)"] \arrow[d, mapsto, "(f\otimes g)\otimes h"] & \sum_i \lambda_i\,(l_i\otimes(m_i\otimes n_i)) \arrow[d, mapsto, "f\otimes(g\otimes h)"] \\
\sum_i \lambda_i\,((f(l_i)\otimes g(m_i))\otimes h(n_i)) \arrow[r, mapsto, "\alpha(A',B',C')"'] & \sum_i \lambda_i\,(f(l_i)\otimes(g(m_i)\otimes h(n_i)))
\end{tikzcd}$$

commutes, and therefore $\alpha$ is an associativity constraint. So 3 is shown.

Recall that the pentagon axiom is satisfied iff for all $R$-modules $K,L,M,N\in Ob(R\mathbf{Mod})$, the diagram

$$\begin{tikzcd}
((K \otimes L) \otimes M) \otimes N \arrow[rr, "\alpha(K,L,M)\otimes \mathrm{id}_N"'] \arrow[d, "\alpha(K\otimes L,M,N)"'] & & (K \otimes (L \otimes M)) \otimes N \arrow[d, "\alpha(K,L\otimes M,N)"] \\
(K \otimes L) \otimes (M \otimes N) \arrow[dr, "\alpha(K,L,M\otimes N)"'] & & K \otimes ((L \otimes M) \otimes N) \arrow[dl, "\mathrm{id}_K\otimes\alpha(L,M,N)"] \\
& K \otimes (L \otimes (M \otimes N)) &
\end{tikzcd}$$

commutes. Then suppose we have $\sum_i \lambda_i \left[((k_i \otimes l_i) \otimes m_i) \otimes n\right]$ in $((K \otimes L) \otimes M) \otimes N$. Then it immediately transpires that the diagram

$$\begin{tikzcd}
\sum_i \lambda_i \left[((k_i \otimes l_i) \otimes m_i) \otimes n\right] \arrow[rr, mapsto, "\alpha(K,L,M)\otimes \mathrm{id}_N"'] \arrow[d, mapsto, "\alpha(K\otimes L,M,N)"'] & & \sum_i \lambda_i \left[(k_i \otimes (l_i \otimes m_i)) \otimes n\right] \arrow[d, mapsto, "\alpha(K,L\otimes M,N)"] \\
\sum_i \lambda_i \left[(k_i \otimes l_i) \otimes (m_i \otimes n)\right] \arrow[dr, mapsto, "\alpha(K,L,M\otimes N)"'] & & \sum_i \lambda_i \left[k_i \otimes ((l_i \otimes m_i) \otimes n)\right] \arrow[dl, mapsto, "\mathrm{id}_K\otimes\alpha(L,M,N)"] \\
& \sum_i \lambda_i \left[k_i \otimes (l_i \otimes (m_i \otimes n))\right] &
\end{tikzcd}$$

commutes, and therefore the pentagon axiom is satisfied. So 4. is shown.

Recall that to say that $l, r$ are left and right unit constraints, respectively, is to say that for all $R$-module homomorphisms $f : N \to M$ the diagrams

$$\begin{tikzcd}
R \otimes N \arrow[r, "l(N)"'] \arrow[d, "\mathrm{id}_I\otimes f"'] & N \arrow[d, "f"] \\
R \otimes M \arrow[r, "l(M)"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
N \otimes R \arrow[r, "r(N)"'] \arrow[d, "f\otimes\mathrm{id}_I"'] & N \arrow[d, "f"] \\
M \otimes R \arrow[r, "r(M)"'] & M
\end{tikzcd}$$

commute, and $l(N), l(M), r(N), r(M)$ are isomorphisms. Suppose $\sum_i \lambda_i(a_i \otimes n_i) \in R \otimes N$ and $\sum_i \lambda_i(n_i \otimes a_i) \in N \otimes R$. Then immediately, we see that the diagrams

$$\begin{tikzcd}
\sum_i \lambda_i(a_i \otimes n_i) \arrow[r, mapsto, "l(N)"'] \arrow[d, mapsto, "\mathrm{id}_I\otimes f"'] & \sum_i \lambda_i a_i n_i \arrow[d, mapsto, "f"] \\
\sum_i \lambda_i(a_i \otimes f(n_i)) \arrow[r, mapsto, "l(M)"'] & \sum_i \lambda_i a_i f(n_i)
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum_i \lambda_i(n_i \otimes a_i) \arrow[r, mapsto, "r(N)"'] \arrow[d, mapsto, "f\otimes\mathrm{id}_I"'] & \sum_i \lambda_i a_i n_i \arrow[d, mapsto, "f"] \\
\sum_i \lambda_i(f(n_i) \otimes a_i) \arrow[r, mapsto, "r(M)"'] & \sum_i \lambda_i a_i f(n_i)
\end{tikzcd}$$

commute. Therefore 5. and 6. are shown.

Recall that here, the triangle axiom is satisfied iff for all $R$-modules $N, M$, the diagram

$$\begin{tikzcd}
(N \otimes R) \otimes M \arrow[rr, "\alpha(N,R,M)"] \arrow[dr, "r_N\otimes\mathrm{id}_M"'] & & N \otimes (R \otimes M) \arrow[dl, "\mathrm{id}_N\otimes l_M"] \\
& N \otimes M &
\end{tikzcd}$$

commutes. Indeed, suppose $\sum_i \lambda_i \left((n_i \otimes a_i) \otimes m_i\right) \in (N \otimes R) \otimes M$. We have that the diagram

$$\begin{tikzcd}
\sum_i \lambda_i \left((n_i \otimes a_i) \otimes m_i\right) \arrow[rr, mapsto, "\alpha(N,R,M)"] \arrow[dr, mapsto, "r_N \otimes \mathrm{id}_M"'] & & \sum_i \lambda_i \left(n_i \otimes (a_i \otimes m_i)\right) \arrow[dl, mapsto, "\mathrm{id}_N \otimes l_M"] \\
& \sum_i \lambda_i a_i \left(n_i \otimes m_i\right) &
\end{tikzcd}$$

commutes. So condition 7 is shown. Therefore we conclude that we have a tensor category. $\square$

In subsequent sections, when we deal with commutative rings, we will not concern ourselves with the field $k$, and assume that $R = k$, and $\varphi : k \to R$ is identity.

# 5 Tangles

In this section, we give a detailed treatment of the category of tangles, and its tensor product, giving definitions and proofs. We develop the definition of a tangle as a set of oriented polygonal arcs residing in a real 3 dimensional space $\mathbb{R}^3$ (which we view as an subset of our universe $\mathcal{U}$). We will see that we are able to compose these tangles, and the set of tangles along with this compositional structure defines a category. It will also be shown that an example of a tensor product of these tangles exists and this tensor operator along with the tensor category defines a monoidal category.

When dealing with tangles, providing explicit proofs for every detail is very cumbersome, unnecessarily repetitive and perhaps even obfuscating, we will often provide an example proof and give proof sketches for similar propositions.

## 5.1 Polygonal Arcs and Links

Knots, links, or tangles of interest are those which are formed in the context of physically moving particles around in physical space.

We consider what might be what first comes to mind when defining a knot. A knot could be defined as a continuous function

$$f : [0,1] \to \mathbb{R}^3$$

such that $f(0) = f(1)$ [20, page 14]. Then what the function does is to trace a path in $\mathbb{R}^3$. Then we might call such a path a knot.

**Definition 88.** Adapted from [20, page 14]. (Preliminary definition of a knot equivalence)

Given continuous functions $f, g : [0,1] \to \mathbb{R}^3$ such that $f(0) = f(1)$ and $g(0) = g(1)$, we shall say that $f$ and $g$ are equivalent, denoted $f \sim g$ iff there exists continuous $\gamma : [0,1] \times [0,1] \to \mathbb{R}^3$ such that

$$\gamma(x, 0) = f(x)$$

$$\gamma(x, 1) = g(x)$$

for all $x \in [0,1]$.

However this turns out to not be ideal. Firstly, it allows the existence of "wild knots," which have erratic behavior. Secondly, any knot is equivalent to the unknot $S^1$. The details of these problems are discussed in [20, page 14].

To circumvent these difficulties, we here introduce the notion of polygonal arcs, adopted from Kassel.

Given two points $M, N \in \mathbb{R}^3$, denote $[M, N]$ as the set

$$\left\{x \in \mathbb{R}^3 \mid \exists \lambda \in [0,1] : x = \lambda M + (1-\lambda)N\right\}$$

and similarly denote $]M, N[$ as the set

$$\left\{x \in \mathbb{R}^3 \mid \exists \lambda \in \,]0,1[\, : x = \lambda M + (1-\lambda)N\right\}$$

**Definition 89.** [12, page 242] A non-empty subset of $L$ of $\mathbb{R}^3$, we state that $L$ is a polygonal arc iff there exist an $n$-tuple $(M_1, ..., M_n)$, where $M_i \in \mathbb{R}^3$ such that:

$$L = \bigcup_{i=1}^{n-1} [M_i, M_{i+1}]$$

$$\forall i, j; i \neq j \implies \;]M_i, M_{i+1}[\; \cap \;]M_j, M_{j+1}[\; = \varnothing; \text{ and } M_i \neq M_j$$

Such an $n$-tuple $(M_1, ..., M_n)$ is said to be an "ordered set of vertices of $L$". We will equivalently say "$(M_1, ..., M_n)$ generates the polygonal arc $L$". We say that $L$ is "closed" iff $M_1 = M_n$. We note that each $M_i$ and $M_j$ must be distinct unless $i, j \in \{1, n\}$.

**Definition 90.** [12, page 242] We define $\partial(L) = \{M_1, M_n\}$ if open, and $\partial(L) = \varnothing$ if closed. The set $\partial L$ is said to be the "boundary" of $L$.

**Example 91.** $L = [(0,0,0),(0,0,1)] \cup [(0,0,1),(0,1,1)]$ is a simple open polygonal arc with boundary $(0,0,0)$ and $(0,1,1)$. This is illustrated in figure 3.

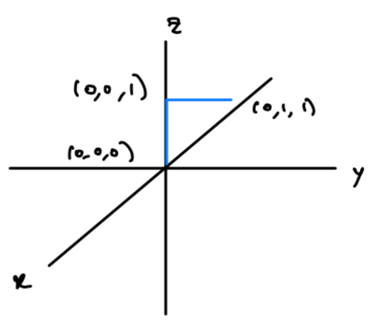


Figure 3: Example of an open polygonal arc.

**Example 92.** $L = [(0,0,0),(0,0,1)] \cup [(0,0,1),(0,1,1)] \cup [(0,1,1),(0,1,0)] \cup [(0,1,0),(0,0,0)]$ is a simple closed polygonal arc. An example of an ordered set of vertices for $L$ is

$$((0,0,0),(0,0,1),(0,1,1),(0,1,0),(0,0,0))$$

This is illustrated in figure 4.

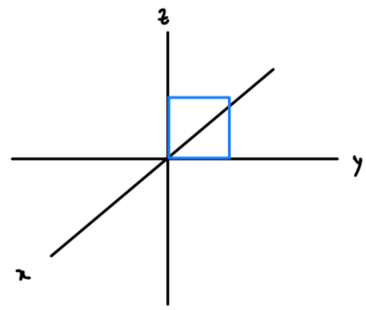


Figure 4: Example of a closed polygonal arc.

**Definition 93.** [20, page 16] For arcs $L = \bigcup_{i=1}^{n-1} [M_i, M_{i+1}]$ and $L' = \bigcup_{i=1}^{m-1} \left[M_i', M_{i+1}'\right]$, we shall say that they are related by a $\Delta$-move iff $m = n + 1$ and there exists some $i$ such that for all $j < i$: $M_j = M_j'$ and for all $j \geq i$: $M_j = M_{j+1}'$. This is illustrated in figure 5.

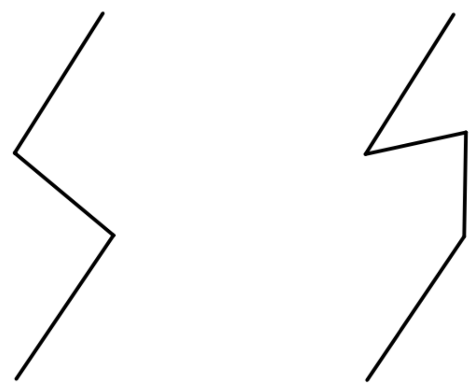

Figure 5: Application of a $\Delta$-move to a polygonal arc.

That is, a $\Delta$-move corresponds introducing a point into an arc.

## 5.2 Orientation of Arcs

Kassel [12, page 242] mentions that links can be oriented. In this section, I propound a rigorous framework with which to define the orientation of polygonal arcs.

We first introduce the following definition, which will be useful in various settings, such as in defining the "stick number" of a link or knot. The stick number is mentioned but not defined in [20, page 27].

**Definition 94.** Recalling definition 89 of an ordered set of vertices, denote $Vert(L)$ as the set

$$\left\{ (M_1, ..., M_n) \in \bigcup_{i \in \mathbb{Z}_+} (\mathbb{R}^3)^i \mid (M_1, ..., M_n) \text{ is an ordered set of vertices of } L; n \in \mathbb{Z}_+ \right\}$$

Any element of $Vert(L)$ is said to "generate $L$".

**Definition 95.** The "stick number of $L$" is defined as the infimum of the non-empty set

$$NVert(L) = \{n \in \mathbb{Z}_+ \mid n+1 \text{ is the length of } x \text{ for some } x \in Vert(L)\}$$

which exists due to well-order.

**Example 96.** Given $L = [(0,0,0),(0,0,1)]$, an element in $Vert(L)$ may be $((0,0,0),(0,0,1))$. Another element may be $((0,0,0),(0,0,1/2),(0,0,1))$. Its stick number is 1 because the shortest sequence of points that can generate it is 2.

**Definition 97.** For $n$-tuples generating closed links, we shall say that $(N_1,...,N_n)$ is a "cyclic reordering" of $(M_1,...,M_n)$ iff

$$(N_1,...,N_n) = (M_i, M_{i+1},...,M_{n-1},M_1,...,M_i)$$

**Example 98.** A cylic reordering of the sequence

$$((0,0,0),(0,0,1),(0,1,1),(0,1,0),(0,0,0))$$

which generates the closed polygonal arc

$$L = [(0,0,0),(0,0,1)] \cup [(0,0,1),(0,1,1)] \cup [(0,1,1),(0,1,0)] \cup [(0,1,0),(0,0,0)]$$

may be

$$((0,0,1),(0,1,1),(0,1,0),(0,0,0),(0,0,1))$$

.

**Proposition 99.** *Denote $\mu$ as the number of sticks for polygonal arc $L$. If $L$ is closed, then there exists up to reordering and reflection, only one $(M_1,...,M_n)$ such that $n=\mu$. If $L$ is open, then there exists up to reflection, only one $(M_1,...,M_n)$ such that $n=\mu$.*

*Proof.* We give here only a sketch of a slightly long proof.

First note that two consecutive points cannot be equal in $(M_1,...,M_n)$, otherwise we can delete one of them. Suppose $(M_1,...,M_n)$ and $(N_1,...,N_n)$ generate the same polygonal arc.

Suppose $M_n = M_1$. We have that $\bigcup_{i=1}^{n-1}[N_i,N_{i+1}] \cap \bigcup_{i=1}^{n-1}[M_i,M_{i+1}]$ is non-empty, so $M_1 \in \bigcup_{i=1}^{n-1}[N_i,N_{i+1}]$. Now suppose that $M_1 \notin \{N_1,...,N_n\}$. Then $M_1 \in ]N_i,N_{i+1}[$ for some $i$. Then $M_2$ must be in $[N_i,N_{i+1}]$ otherwise $[M_1,M_2]$ and $[N_i,N_{i+1}]$ will not be the same line segment and because a polygonal arc cannot go through itself, we obtain a contradiction.

Then by similar reasoning, $M_{n-1} \in [N_j,N_{j+1}]$. But then $M_n = M_1$ is therefore redundant, and we can write

$$L = \bigcup_{i=2}^{n-1}[M_i,M_{i+1}]$$

a contradiction. So $M_1 = N_j$ for some $j$.

Then $M_2$ is equal to the next or previous distinct point in the sequence $(N_1,...,N_{n-1})$ otherwise we end up with the same contradiction and so on, until we obtain equality up to cyclic reordering and reflection.

Conversely, suppose $M_n \neq M_1$. If $n = 2$, the statement is obvious. Suppose the statement holds for $n - 1 \geq 1$. Suppose $(N_1, ..., N_n)$ is another $n$-tuple such that

$$L = \bigcup_{i=1}^{n-1} [N_i, N_{i+1}]$$

$$\forall i, j; i \neq j \implies \ ]N_i, N_{i+1}[\ \cap\ ]N_j, N_{j+1}[\ = \varnothing$$

Then $M_1 = N_1$ or $M_1 = N_n$ otherwise $\partial(L) = \{M_1, M_n\} \neq \{N_1, N_n\} = \partial(L)$. Suppose $M_1 = N_1$.

Then $(M_2, ..., M_n)$ and $(N_2, ..., N_n)$ an ordered set of vertices for some $L'$ whose stick number is $n - 1$. By the induction hypothesis,

$$(M_2, ..., M_n) = (N_2, ..., N_n)$$

or

$$(M_2, ..., M_n) = (N_n, ..., N_2)$$

If the latter case holds, then

$$(M_1, ..., M_n) = (N_1 N_n, ..., N_2)$$

and so $(N_1 N_n, ..., N_2)$ and $(N_1, ..., N_n)$ are the vertices of the same arc. Now close this arc; we have that $(N_1 N_n, ..., N_2, N_1)$ and $(N_1, ..., N_n, N_1)$ generate the same closed arc. But then by definition of an arc generated by $(N_1 N_n, ..., N_2, N_1)$, we have that $]N_1, N_n[$ is disjoint from $]N_i, N_{i+1}[$ for all $i < n$. So $]N_1, N_n[$ intersect with the polygonal arc generated by $(N_1, ..., N_n)$ is empty, which means that the arcs generated by $(N_1 N_n, ..., N_2)$ and $(N_1, ..., N_n)$ are not the same, a contradiction. So

$$(M_2, ..., M_n) = (N_2, ..., N_n)$$

hence

$$(M_1, ..., M_n) = (N_1, ..., N_n)$$

Otherwise, suppose $M_1 = N_n$. Then by the exact same process as above, we have

$$(M_2, ..., M_n) = (N_{n-1}, ..., N_1)$$

and thus $(M_1, ..., M_n) = r$.

Therefore by induction, the statement holds for all $n$. □

**Definition 100.** The unique element $(M_1, ..., M_n)$ as specified in the previous proposition will be called an "orientation" of a polygonal arc. Up to cyclic reordering, there exists exactly one orientation for each closed polygonal arc. An arc is said to be oriented iff its orientation is specified. In particular, we state that a pair $(L, \omega)$ is an oriented arc iff $L$ is a polygonal arc and $\omega$ is an orentation of $L$. We may write $L$ to denoted orented arc $(L, \omega)$, for some orentation $\omega$. Given arc $L$ and its orientation $\omega = (M_1, ..., M_n)$, we say that $M_1$ is the origin, and $M_n$ is the endpoint of $L$. So $L$ goes from $M_1$ to $M_n$.

*Remark* 101. An orientation $\omega = (M_1, ..., M_n)$ of $L$ is an ordered set of verties of $L$ and generates $L$. Therefore when $L$ is open, $\partial(L)$ can also be regarded as the the first and last element of sequence $\omega$.

An oriented arc $(L, \omega)$ might be denoted as $L$ itself as long as it is understood that $L$ has some orientation to it, as is done in [12]. For two disjoint oriented arcs $(L, \omega), (L', \omega')$ can also write $L \cup L'$ to denote

$$\{(L, \omega), (L', \omega')\}$$

In any case, oriented arcs are understood to be their arcs, bestowed with some orientation.

**Example 102.** For

$$L = [(0,0,0),(0,0,1)] \cup [(0,0,1),(0,1,1)] \cup [(0,1,1),(0,1,0)] \cup [(0,1,0),(0,0,0)]$$

its two unique oreintations is determined by the sequences

$$((0,0,0),(0,0,1),(0,1,1),(0,1,0),(0,0,0))$$

whose corresponding illustration is given in the left picture in figure 6, and

$$((0,0,0),(0,1,0),(0,1,1),(0,0,1),(0,0,0))$$

whose corresponding illustration is given in the right picture in figure 6.

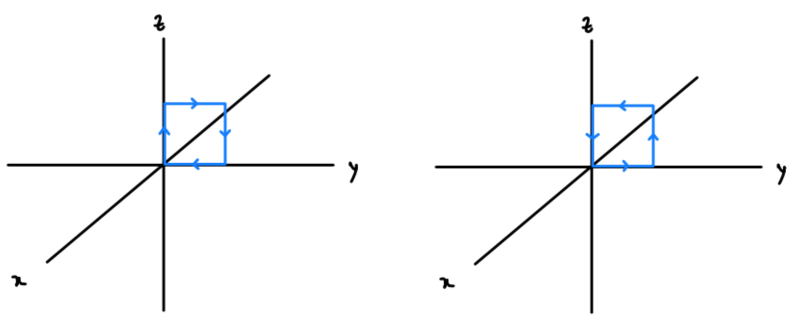


Figure 6: The two orientations of a closed polygonal arc.

### 5.3 Disjoint Union of Oriented Polygonal Arcs

**Definition 103.** We say that a set $L$ is a "disjoint union of oriented polygonal arcs" iff it is expressed as $L = \bigcup_{i \in I}\{(L_i, \omega_i)\}$, such that $L_i \cap L_j = \varnothing$ if $i \neq j$, where $(L_i, \omega_i)$ are polygonal arcs. It is further said to be finite iff the index set $I$ is finite. In this paper, we will always assume $I$ to be finite.

**Definition 104.** Given disjoint union of oriented polygonal arcs $L = \bigcup_{i \in I}\{(L_i, \omega_i)\}$, we define the boundary $\partial(L) = \bigcup_i \partial(L_i)$.

We will briefly talk about links and knots before developing tangles.

**Definition 105.** A set $L$ is said to be a "link" iff it is expressed as a disjoint union of a finite number of closed oriented polygonal arcs.

**Example 106.** Figure 7 shows two examples of links.

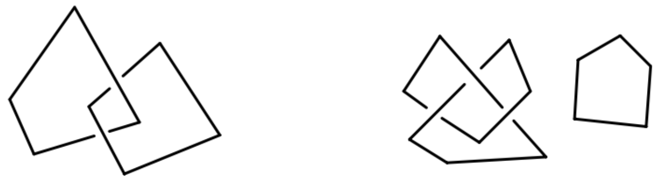

Figure 7: Two examples of links

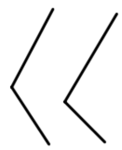

Figure 8: A non link

**Example 107.** Figure 8 shows an example of a non link.

**Proposition 108.** *If $L$ is expressed as a union of $m$ disjoint polygonal arcs, and is also expressed as a union of $n$ disjoint polygonal arcs, then $m = n$.*

The proposition above, which we will not provide a proof of, allows us to define the "order" of a link.

**Definition 109.** [12, page 242] The number of polygonal arcs that make up a link is said the be its "order". A link of order 1 is called a "knot".

## 5.4 Joining Oriented Polygonal Arcs

When we have two strings, we can tie the ends together to form a longer string. The same concept is applied here; if the end of one polygonal arc is the beginning of another, then we can join them to form one longer polygonal arc.

**Definition 110.** Given two open polygonal arcs $(L, \omega), (L', \omega')$, where $\omega = (M_1, ..., M_n)$ and $\omega' = (M_1', ..., M_m')$, we will state that $(L, \omega), (L', \omega')$ are "composible" iff

1. $\{M_n\} \subset L \cap L' \subset \{M_1, M_n\}$.

2. If the union $L \cup L'$ is a polygonal arc.

From the above definition, we see that $(M_1, ..., M_n, M_2', ..., M_m')$ is an ordered set of vertices of $L \cup L'$. We want to define the orientation of $L \cup L'$ as the orientation which starts at $M_1$ and ends at $M_m'$.

**Example 111.** For $L, L'$, where

$$L = [(0, 0, 1), (0, 0, 1/2)]$$

with orientation

$$((0, 0, 1), (0, 0, 1/2))$$

and

$$L' = [(0,0,1/2),(0,0,0)]$$

with orientation

$$[(0,0,1/2),(0,0,0)]$$

Then we have that

1. $\{(0,0,1/2)\} \subset L \cap L' = \{(0,0,1/2)\} \subset \{(0,0,1),\{(0,0,1/2)\}$.
2. $L \cup L' = [(0,0,1),(0,0,0)]$, so it is a polygonal arc.

So it is composible.

**Definition 112.** Given two open polygonal arcs $(L,\omega),(L',\omega')$, where $\omega = (M_1,...,M_n)$ and $\omega' = (M'_1,...,M'_m)$, if the union $L \cup L'$ is a polygonal arc, and $M_n = M'_1$, then the join of $(L,\omega)$ and $(L',\omega')$, denoted $(L',\omega') \circ (L,\omega)$, is defined as

$$(L \cup L', \omega'')$$

where $\omega''$ is the orientation of $L \cup L'$ such that $M_1$ is its first element and $M'_m$ is its last element up to cyclic reordering.

Note that if $M_1 = M'_m$ as well, then $\omega''$ is not unique, but it has a cyclic reordering such that the condition above is satisfied.

**Example 113.** Figure 9 shows the composition of two polygonal arcs.

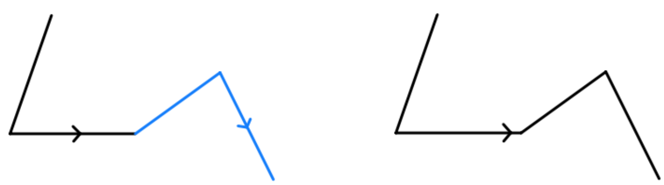

Figure 9: The left drawing shows two composible arcs. The right is the result of their composition.

**Proposition 114.** *The composition of polygonal arcs is a polygonal arc.*

*Proof.* This is by definition. □

**Definition 115.** Given two disjoint union of oriented polygonal arcs $L = \bigcup_{i \in I}\{(L_i,\omega_i)\}$, $L' = \bigcup_{j \in J}\{(L'_j,\omega'_j)\}$ we will state that they "are composible" iff for any two links $(L_i,\omega_i)$ and $(L'_j,\omega'_j)$, if $L_i \cap L'_j \neq \varnothing$, then $(L_i,\omega_i)$ are $(L'_j,\omega'_j)$ composible.

**Example 116.** Figure 10 shows two examples of composible arcs.

*Remark* 117. It follows from the above definition that for any $(L_i,\omega_i)$, there exists at most only one $(L'_j,\omega'_j)$ such that $(L_i,\omega_i)$ are $(L'_j,\omega'_j)$ composible. This is because $L'_i \cap L'_j = \varnothing$ if $i \neq j$ in $J$. The same holds for any $(L'_j,\omega'_j)$.

Figure 10: The drawing on the left shows a set of black tangles and a set of one blue tangle. Their composition is simply the union of these arcs. On the other hand, the drawing on the right has a non-zero intersection between black and blue tangles. In this case, we need to compose the intersecting arcs to obtain the composition of tangles. The result is shown in figure 11.

**Definition 118.** Given two oriented polygonal arcs $L = \bigcup_{i \in I}\{(L_i, \omega_i)\}$, $L' = \bigcup_{j \in J}\{(L'_j, \omega'_j)\}$ which are disjoint and composible, we define its join or composition, denoted $L' \circ L$, as the union of the three sets

$$\{(L_i, \omega_i) \mid (L_i, \omega_i) \text{ is not composible with any } (L'_j, \omega'_j)\}$$

$$\{(L'_j, \omega'_j) \mid (L'_j, \omega'_j) \text{ is not composible with any } (L_i, \omega_i)\}$$
$$\{(L'_j, \omega'_j) \circ (L_i, \omega_i) \mid (L_i, \omega_i) \text{ and } (L'_j, \omega'_j) \text{ are composible}\}$$

**Example 119.** Figure 11 shows the composition two sets of arcs.

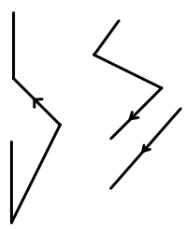

Figure 11: Composition of two sets of arcs in the left drawing in figure 10.

### 5.5 Tangles

In the following diagrams we will draw polygonal arcs as smoothed out lines, because they are easier to draw. Formally, however, they need to be made of straight lines only.

We start by first giving a drawing of an example of a tangle.

**Example 120.** Figure 12 shows an example of a tangle.

**Example 121.** Figure 13 shows an example of a non-tangle.

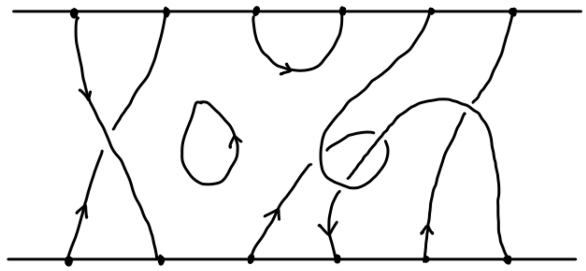

Figure 12: Example of a tangle.

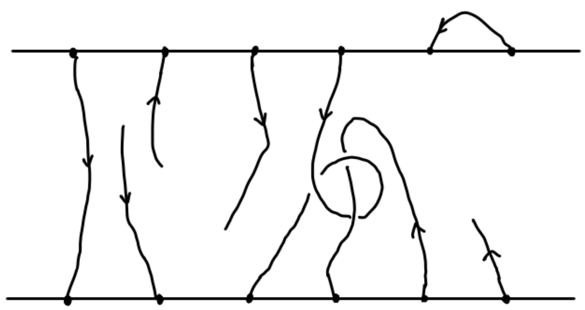

Figure 13: Example of a non-tangle.

We shall denote the set $\{1, 2, \ldots, n\}$ by writing $[n]$ as does Kassel [12, page 257]. When we write $[0]$ we mean the empty set.

We present a definition of the category of tangles from Kassel [12]. The first axis will be called the $x$ axis; the second axis the $y$ axis, and the third the $z$ axis. We affix an ordered set of points, from which tangles originate and another ordered set of points at which they end.

**Definition 122.** Adapted from [12, page 257, 258]. A set $T$ is said to be a tangle iff:

1. $T = \bigcup_i \{(L_i, \omega_i)\}$ ; it is a disjoint union of oriented polygonal arcs $(L_i, \omega_i)$, each of the arcs being a subset of $\mathbb{R}^2 \times [0, 1]$.

2. There exists pair $(k, l) \in \mathbb{N}^2$ such that $\partial(T) = T \cap \mathbb{R}^2 \times \{0, 1\} = [k] \times \{(0, 0)\} \cup [l] \times \{(0, 1)\}$.

*Remark* 123. The choice of making $[k] \times \{(0, 0)\} \cup [l] \times \{(0, 1)$ the start and end points of tangles is arbitrary. I note here that a more general (and mathematically elegant) construction would consider an numbered collection of $k$ and $l$ arbitrary points in $\mathbb{R}^3$ space, and define an equivalence class of points to be the start or end of a tangle, but this is unnecessarily complicated with little insight to gain.

**Definition 124.** Here $T \cap \mathbb{R}^2$ is written to mean $(\bigcup_i L_i) \cap \mathbb{R}^2$, where $(L_i, \omega_i)$ are the oriented arcs of $T$.

**Example 125.** A link is a tangle of type $(0, 0)$. In particular, a knot is a tangle of type $(0, 0)$.

The pair $(k, l)$ is clearly unique, and is said to be the "type" of $T$. That is, we say that $T$ is a tangle of type $(k, l)$. Note that we require to check both equalities in condition 3. The origin and endpoints of each polygonal arc needs to be in the form $(n, 0, 0)$ or $(n, 0, 1)$.

So will say that 1 is "up" and 0 is "down" in the $z$-direction.

**Definition 126.** [12, page 258] Given a tangle $T$ of type $(k, l)$, we shall define $\sigma(T)$ and $\tau(T)$ as follows

$$\sigma(T) = \begin{cases} \varnothing & k = 0 \\ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) & k \neq 0 \end{cases} \qquad \tau(T) = \begin{cases} \varnothing & l = 0 \\ (\eta_1, \eta_2, \ldots, \eta_k) & l \neq 0 \end{cases}$$

where, symbollically, we put $\varepsilon_i = +$ whenever $(i, 0, 0) \in T$ and it is the final element in the orientation $\omega$ of $T$. That is to say, it is the endpoint of $T$. We put $\varepsilon_i = -$ otherwise.

We put $\eta_i = +$ whenever $(i, 0, 1) \in T$ and it is the first element in the orientation $\omega$ of $T$. That is to say, it is the origin of $T$. We put $\eta_i = -$ otherwise.

**Example 127.** [12, page 258] The link $[(1, 0, 1), (1, 0, 0)]$ with orientation $((1, 0, 1), (1, 0, 0))$ is denoted by the symbol $\downarrow$ and is drawn in figure 14.

**Example 128.** [12, page 258] The link $[(1, 0, 0), (1, 0, 1)]$ with orientation $((1, 0, 0), (1, 0, 1))$ is denoted by the symbol $\uparrow$ and is drawn in figure 14.

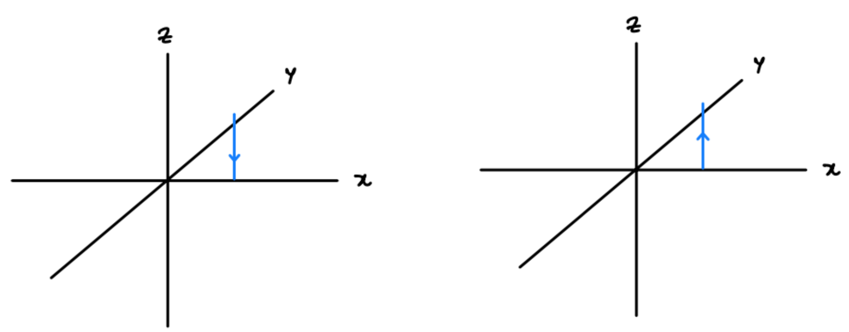


Figure 14: The left diagram shows $\downarrow$ and the right diagram shows $\uparrow$

### 5.6 Equivalence of Tangles

From now on, when we have a function $f : \mathbb{R}^2 \times \{0, 1\} \to \mathbb{R}^2 \times \{0, 1\}$, and tangle $T = \bigcup_i \{(L_i, \omega_i)\}$, we will write $f[T]$ to denote $\bigcup_i \{(f[L_i], f[\omega_i])\}$. That is, when applying a transformation to a tangle, we transform both the lines of the tangle along with its associated orientation, so that the $f[\omega_i]$ gives an orientation of $f[L_i]$, if $f[L_i]$ is an arc.

**Definition 129.** Adapted from [12, page 242] For $U \subset \mathbb{R}^n$, we say that a continuous map $f : U \to \mathbb{R}^m$ is "piecewise linear" iff there exists a partition of $U_i$ of $U$ such that the restriction of $f$ on $U_i$ coincides with the map

$$x \mapsto Mx + b$$

where $M$ is some $n \times m$ matrix and $b$ is a vector of dimension $m$. That is, it is affine on $U_i$.

*Remark* 130. Since a piecewise linear function is defined piecewise with affine functions, we might be inclined to call them piecewise affine functions. However, this is not the official name that they are given.

*Remark* 131. We note that the composition of affine maps is an affine map.

Given two variable function $f$, we will denote the map $f(\bullet, x)$ to denote the map

$$f(\bullet, x)(y) := f(y, x)$$

.

Given a tangle that is a subset of $(\mathbb{R}^2 \times [0,1])$, we can consider the moving around of these tangles with respect to a time parameter $t$. We parameterize this on the interval $[0,1]$. If a tangle can be continuously moved so that it becomes another tangle, then we would say that these tangles are the same.

**Definition 132.** Adapted from [12, page 243]. A map $f : (\mathbb{R}^2 \times [0,1]) \times [0,1] \to (\mathbb{R}^2 \times [0,1])$ is said to be a "polygonal isotopy of tangles" (or in short, "isotopy") iff

1. It is a piecewise linear map.
2. $f(\bullet, t)$ is a homeomorphism for all $t \in [0,1]$.
3. $f(\bullet, t)$ restricted on the set $(\mathbb{R}^2 \times \{0,1\})$ is the identity map for all $t \in [0,1]$.
4. $f(\bullet, 0) : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ is the identity map.

*Remark* 133. Note that we do not need to introduce the notion of preservation of orientation as the origin and endpoint of any polygonal arc is always kept the in the same place throughout the isotopy. So orientation is always preserved.

**Example 134.** The empty set, which is a tangle of no polygonal arcs, is only isotopic to itself under the identity map $\mathrm{id} : (\mathbb{R}^2 \times [0,1]) \times [0,1] \to (\mathbb{R}^2 \times [0,1])$.

We will state the following useful result without proof.

**Proposition 135.** *Two tangles are equivalent iff they are related by a sequence of $\Delta$-moves.*

**Definition 136.** When we give a function $f$, and $T = \bigcup_i \{(L_i, (M_1, \cdots, M_{n(i)}))\}$ is union of disjoint polygonal arcs, we will write $f[T]$ to denote the set

$$\bigcup_i \{(f[L_i], (f(M_1), \cdots, (M_{n(i)})))\}$$

**Proposition 137.** *If $T$ is a tangle of type $(k,l)$, and $f$ is an isotopy, then $f(\bullet, t=1)[T]$ is a tangle of type $(k,l)$.*

*Proof.* To say that $T$ is of type $(k,l)$ is to say that $\partial(T) = T \cap \mathbb{R}^2 \times \{0,1\} = [k] \times \{(0,0)\} \cup [l] \times \{(0,1)\}$. By condition 2 and 3 of definition 132, we obtain that $f(\bullet, t=1)[T] \cap \mathbb{R}^2 \times \{0,1\} = T \cap \mathbb{R}^2 \times \{0,1\}$. Further, due to remark 101, and the fact that the first and last element of the orientation sequence $\omega = (M_1, \cdots, M_n)$ for some any arc is unchanged due to condition 3 of definition 132 we have that $\partial(f(\bullet, t=1)[T]) = \partial(T)$. Therefore $f(\bullet, t=1)[T]$ is of same type as that of $T$. $\square$

**Definition 138.** Given equivalence class of tangles $[T]$, we define its type as the type of a representative element $T$.

**Definition 139.** Inspired by [12, page 243]. Two tangles $T$ and $T'$ are said to be isotopic iff there exists polygonal isotopy $f$ such that $f(\bullet, 1)[T] = T'$. Further, when $f$ is a polygonal isotopy such that $f(\bullet, 1)[T] = T'$, we shall say that "$f$ is an isotopy from $T$ to $T'$".

Kassel [12, page 243] mentions the following fact, to which we provide a proof.

**Proposition 140.** *Write $T \sim T'$ iff $T$ is isotopic to $T'$. Then $\sim$ is an equivalence relation.*

*Proof.* We have that $T \sim T$ because the desired isotopy is exhibited by the identity map of $\mathbb{R}^2 \times [0, 1]$.

Suppose $T \sim T'$. Then take polygonal isotopy $f$. Take partition $U_i \subset \mathbb{R}^n \times [0, 1]$ and $2 \times 2$ matrix $M_i$, and 2-dimensional vectors $m_i, b_i$ such that

$$f\mid_{U_i}(x, t) = M_i x + m_i t + b_i$$

for $x \in U_i$. Since $f(\bullet, t)$ is a homeomorphism for all $t \in [0, 1]$, we have that $M_i$ must be invertible for all $i$.

Define $g : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^n$ by putting

$$g(x, t) = M_i^{-1} x - M_i^{-1} m_i t - M_i^{-1} b_i$$

for $x \in U_i$.

Then

1. $g$ is piecewise linear (by defininition).

2. $g(\bullet, t)$ is in fact the inverse of $f(\bullet, t)$. On $U_i$, we have

$$g(\bullet, t) \circ f(\bullet, t)(x) = M_i^{-1}(M_i x + m_i t + b_i) - M_i^{-1} m_i t - M_i^{-1} b_i = x$$

and

$$f(\bullet, t) \circ g(\bullet, t)(x) = M_i(M_i^{-1} x - M_i^{-1} m_i t - M_i^{-1} b_i) + m_i t + b_i = x$$

Having that the inverse of a homeomorphism is a homeomorphism (which is by definition), we have that $g(\bullet, t) =$ is a homeomorphism for all $t \in [0, 1]$.

3. Suppose $(x, t) \in (\mathbb{R}^2 \times \{0, 1\}) \times [0, 1]$. We have that the third component of $x = (x_1, x_2, x_3)$ is 0 or 1. Then on $U_i$, we have $f(x, t) = M_i x + m_i t + b_i = x$. Therefore multiplying $M_i^{-1}$ on both sides, we get $x + M_i^{-1} m_i t + M_i^{-1} b_i = M_i^{-1} x$. So we get $g(x, 0) = M_i^{-1} x - M_i^{-1} m_i t - M_i^{-1} b_i = x$. So we get $g(x, t) = M_i^{-1} x - M_i^{-1} m_i t - M_i^{-1} b_i = x$. So $g(\bullet, t)$ restricted on the set $(\mathbb{R}^2 \times \{0, 1\})$ is the identity map for all $t \in [0, 1]$.

4. On $U_i$, we have $f\mid_{U_i}(x, 0) = M_i x + b_i = x$. So multiplying by $M_i^{-1}$ on both sides, we get $x + M_i^{-1} b_i x = M_i^{-1} x$. So $g(\bullet, 0)(x) = g(x, 0) = M_i^{-1} x - M_i^{-1} b_i = x$. So $g(\bullet, 0) : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map.

So indeed, we see that an isotopy. Further, we have that

$$g(\bullet,1)[T'] = g(\bullet,1)[f(\bullet,1)[T]] = g(\bullet,1)[\{M_i x + m_i + b_i \mid x \in T\}]$$

$$= \left\{M_i(M_i^{-1}x - M_i^{-1}m_i t - M_i^{-1}b_i) + m_i + b_i \mid x \in T\right\} = T$$

and therefore it is an isotopy from $T$ to $T'$. Thus $T' \sim T$.

Now suppose $T \sim T' \sim T''$. Then take $f$ as an isotopy from $T$ to $T'$ and $g$ as an isotopy from $T'$ to $T''$. Then $g : (\mathbb{R}^2 \times [0,1]) \times [0,1] \to (\mathbb{R}^2 \times [0,1])$ defined by putting

$$h(x,t) = g(f(x,t),t)$$

We show that this in fact is an isotopy. For take $U_i \subset (\mathbb{R}^2 \times [0,1])$ and matrices $M_i$, vectors $m_i, t_i$ such that

$$f\mid_{U_i}(x,t) = M_i x + m_i t + b_i$$

and take $U_i' \subset (\mathbb{R}^2 \times [0,1])$ and matrices $M_i'$, vectors $m_i', b_i'$ such that

$$g\mid_{U_i'}(x,t) = M_i' x + m_i' t + b_i'$$

We continue to verify the four conditions.

1. Any element in $(\mathbb{R}^2 \times [0,1])$ is in $U_i \cap U_j'$ for some pair $(i,j)$. On such a domain, we have

   $$h(x,t) = M_i'(M_i x + m_i t + b_i) + m_i' t + b_i'$$

   $$= M_i M_i' x + (M_i' m_i + m_i)t + (b_i + b_i')$$

   and so $h$ is a piecewise linear function.

2. Given fixed $t$, we have $h(\bullet,t)(x) = h(x,t) = g(f(x,t),t) = g(\bullet,t) \circ f(\bullet,t)(x)$. We have that both $g(\bullet,t)$ and $f(\bullet,t)$ are homeomorphisms, and the composition of homeomorphisms is a homeomorphism. So $h(\bullet,t)$ is a homeomorphism for all $t \in [0,1]$.

3. Suppose that $(x,t) \in (\mathbb{R}^2 \times \{0,1\}) \times [0,1]$. Then $h(x,t) = g(f(x,t),t) = g(x,t) = x$. So $h(\bullet,t)$ restricted on the set $(\mathbb{R}^2 \times \{0,1\})$ is the identity map for all $t \in [0,1]$.

4. We have $h(x,0) = g(f(x,0),0) = g(x,0) = x$. So $h(\bullet,0) : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map.

So indeed, $h$ is an isotopy. We finally verify that it takes $T$ to $T''$:

$$h(\bullet,1)[T] = g(f(\bullet,t),t)[T] = g(\bullet,t) \circ f(\bullet,t)[T] = g(\bullet,t)[T'] = T''$$

And therefore we conclude that $T \sim T''$. So $\sim$ is an equivalence. □

**Example 141.** Figure 15 gives two equivalent tangles and a non-equivalent tangle.

We shall therefore consider the equivalence classes of tangles. Denote $[T]$ as the set of all tangles that are equivalent to tangle $T$.

*Remark* 142. We note that, due to condition 3 of definition 132, if $T \sim T'$, then $\sigma(T) = \sigma(T')$ and $\tau(T) \sim \tau(T')$. Therefore, we shall define $\sigma([T]) := \sigma(T)$ and $\tau([T]) := \tau(T)$.

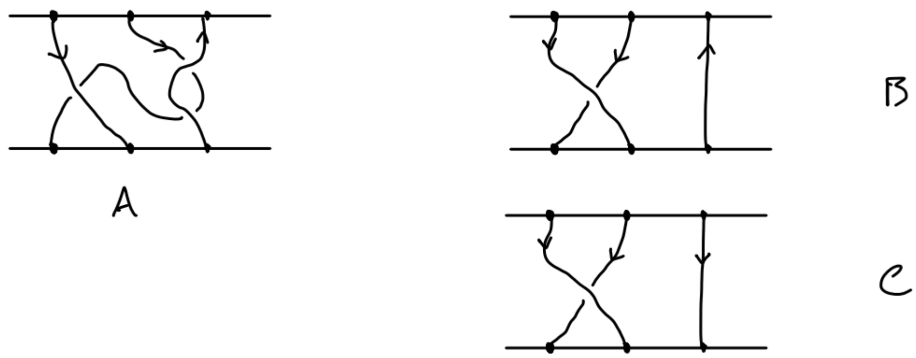


Figure 15: A and B are equivalent, but A and C are not.

### 5.7 Composition of Tangles

Given two tangles $T$ and $T'$, if we have $\sigma(T') = \tau(T)$, we want to as define something similar to the joining of links in definition 118.

**Definition 143.** Given two tangles $T$ and $T'$, we say that they are "composible" iff we have $\sigma(T') = \tau(T)$.

Given tangles $T = \bigcup_{i \in I}\{(L_i, \omega_i)\}$, $T' = \bigcup_{j \in J}\{(L'_j, \omega'_j)\}$, define

$$\overline{L_j} = \{(x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, y, 2u - 1) \in L_j\}$$

$$\overline{\omega_j} = ((x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, y, 2u - 1) \in \omega_j)$$

$$\overline{L'_j} = \{(x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, y, 2u) \in L'_j\}$$

$$\overline{\omega'_j} = ((x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, y, 2u) \in \omega'_j)$$

**Example 144.** Figure 16 shows an example of composible tangles.

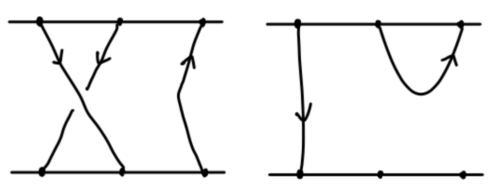

Figure 16: The tangle on the left can be composed with the tangle on the right. The result of their composition is shown in figure 18.

**Example 145.** Figure 17 shows an example of non-composible tangles.

**Proposition 146.** *Then we have that* $\overline{T} := \bigcup_{i \in I}\{(\overline{L_i}, \overline{\omega_i})\}$, $\overline{T'} = \bigcup_{j \in J}\{(\overline{L'_j}, \overline{\omega'_j})\}$ *are composible, and that any open arc in* $\overline{T}$ *is composible with a unique arc in* $\overline{T'}$.

*Proof.* Although we do not give the details, we see that this follows from the fact that $\sigma(T') = \tau(T)$. $\square$

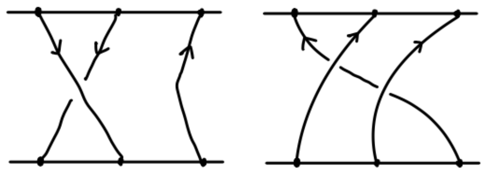

Figure 17: The tangle on the left cannot be composed with the tangle on the right.

**Definition 147.** Given two composible tangles $T = \bigcup_{i\in I}\{(L_i, \omega_i)\}$, $T' = \bigcup_{j\in J}\{(L'_j, \omega'_j)\}$ we define its join, denoted $T' \circ T$, as the arc definition join [see definition 118] of $\overline{T}$ and $\overline{T'}$. In particular, we note that for composible arcs $L_i$ and $L'_i$

$$L'_i \circ L := \{(x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, y, 2u - 1) \in L_i \text{ or } (x, y, 2u) \in L'_i\}$$

and we have that

$$T' \circ T = \bigcup_i \{(\overline{L'_j}, \overline{\omega'_j}) \circ (\overline{L_j}, \overline{\omega_j})\}$$

**Example 148.** As per example 144, their composition is given in figure 18.

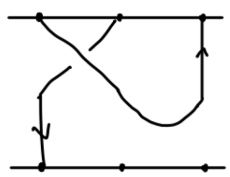

Figure 18: Result of composition of tangles from figure 16.

**Proposition 149.** *If $T$ and $T'$ are composible tangles then $T' \circ T$ is a tangle, and futher, $\sigma(T' \circ T) = \sigma(T)$ and $\tau(T' \circ T) = \tau(T')$.*

*Proof.* We have that $T' \circ T$ is a tangle because

1. By definition it is a union of oriented polygonal arcs $(L_i, \omega_i)$, each of the arcs being a subset of $\mathbb{R}^2 \times [0, 1]$, because composition of arcs is an arc, from proposition 114.

2. If $T$ is of type $(k, l)$ and $T'$ is of type $(k', l')$, we have that $T' \circ T \cap \mathbb{R}^2 \times \{0, 1\} = [l] \times \{(0, 1)\}$, and $T' \circ T \cap \mathbb{R}^2 \times \{0, 0\} = [l'] \times \{(0, 0)\}$. Further, since each arc in $T'$ is joined with an arc in $T$, we have that all the endpoints reside in $T \cap \mathbb{R}^2 \times \{0, 1\}$, and we have $\partial(T) = T \cap \mathbb{R}^2 \times \{0, 1\}$.

The fact that $\sigma(T' \circ T) = \sigma(T)$ and $\tau(T' \circ T) = \tau(T')$ is quite obvious if we look at it geometrically; we simply squeezed the set $T$ to upper half of the interval $[0, 1]$ in the $z$-direction, and $T'$ to lower half of the interval $[0, 1]$ in the $z$-direction. $\square$

**Proposition 150.** *Composition is invariant under equivalence of tangles. That is, if $T \sim T'$ and $U \sim U'$, and $\sigma(U) = \tau(T)$, then $U \circ T \sim U' \circ T'$.*

*Proof.* In this particular case, we give a fully detailed proof for didactic purposes.

Suppose $T \sim T'$ and $U \sim U'$, and $\sigma(U) = \tau(T)$. Then, as previously noted, we have $\sigma(U') = \tau(T')$, so $U' \circ T'$ is defined. We have

$$U' \circ T' := \{(x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, 2u - 1) \in T' \text{ or } (x, 2u) \in U'\}$$

We shall exhibit isotopy from $U \circ T$ to $U' \circ T'$.

Given an $n$-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and function $f : \mathbb{R} \to \mathbb{R}$. we will define

$$\Omega_f(x_1, \ldots, x_n) := (x_1, \ldots, f(x_n))$$

Denote here the function $\nu : \mathbb{R} \to \mathbb{R}$ as the associtation $\nu : x \mapsto x/2$. Also denote here the function $\xi : \mathbb{R} \to \mathbb{R}$ as the associtation $\xi : x \mapsto x/2 + 1/2$.

Note that $\Omega_\nu$ and $\Omega_\xi$ are both affine transformations in $\mathbb{R}^n$. That is, $\Omega_\nu(x) = Ax + b$ and $\Omega_\xi(x) = A'x + b'$ for some matrices $A, A'$ and vectors $b, b'$. Take such elements. We recall that the composition of affine transformations is an affine transformation. The inverse of an affine transformation, if it exists, is also an affine transformation. We have $\Omega_{f^{-1}} = (\Omega_f)^{-1}$, so $\Omega_{\nu^{-1}}$ and $\Omega_{\xi^{-1}}$ are both affine transformations. We also see that $\Omega_f \circ \Omega_g = \Omega_{f \circ g}$.

Take $f : (\mathbb{R}^2 \times [0, 1]) \times [0, 1] \to (\mathbb{R}^2 \times [0, 1])$ and $g : (\mathbb{R}^2 \times [0, 1]) \times [0, 1] \to (\mathbb{R}^2 \times [0, 1])$ such that $g$ is an isotopy from $T$ to $T'$ and $f$ is an isotopy from $U$ to $U'$. Then define $h : (\mathbb{R}^2 \times [0, 1]) \times [0, 1] \to (\mathbb{R}^2 \times [0, 1])$ by putting

$$h(x, t) = \begin{cases} \Omega_\nu f(\Omega_{\nu^{-1}} x, t) & x_3 \in [0, 1/2] \\ \Omega_\xi g(\Omega_{\xi^{-1}} x, t) & x_3 \in [1/2, 1] \end{cases} \tag{2}$$

where we denote $x_3$ as the third component of $x = (x_1, x_2, x_3)$. Then $h$ is an isotopy; we shall verify this. We first note that $h$ is indeed defined on $x_3 = 1/2$, as we have $\Omega_{\nu^{-1}} x = (x_1, x_2, 1)$, which by condition 3 of an isotopy, we get that $f$ does not change the element. So we get

$$\Omega_\nu f(\Omega_{\nu^{-1}} x, t) = \Omega_\nu \Omega_{\nu^{-1}} x = x$$

We also have $\Omega_{\xi^{-1}} x = (x_1, x_2, 0)$, which again by condition 3 of an isotopy, we get that $g$ does not change the element.

$$\Omega_\xi g(\Omega_{\xi^{-1}} x, t) = \Omega_\xi \Omega_{\xi^{-1}} x = x$$

So indeed $h$ is well defined. We continue to verify the four required conditions.

1. Take take $U_i \subset (\mathbb{R}^2 \times [0, 1]) \times [0, 1]$ and matrices $M_i$, vectors $m_i, b_i$ such that

   $$f|_{U_i}(x, t) = M_i x + m_i t + b_i$$

   and take $U_i' \subset (\mathbb{R}^2 \times [0, 1]) \times [0, 1]$ and matrices $M_i'$, vectors $m_i', b_i'$ such that

   $$g|_{U_i'}(x, t) = M_i' x + m_i' t + b_i'$$

   Now define

   $$V_i^{\text{up}} := \{(\Omega_\xi x, t) \mid (x, t) \in U_i'\}$$
   $$V_i^{\text{down}} := \{(\Omega_\nu x, t) \mid (x, t) \in U_i\}$$

Then $\bigcup_i (V_i^{\text{up}} \cup V_i^{\text{down}}) = (\mathbb{R}^2 \times [0,1])$. Suppose $(\Omega_\nu x, t) \in V_i^{\text{down}}$. Then

$$h(\Omega_\nu x, t) = \Omega_\nu f(\Omega_{\nu^{-1}} \Omega_\nu x, t) = \Omega_\nu f\mid_{U_i}(x,t)$$

$$= \Omega_\nu (M_i x + m_i t + b_i) = A(M_i x + m_i t + b_i) + b$$

Suppose $(\Omega_\xi x, t) \in V_i^{\text{up}}$. Then

$$h(\Omega_\xi x, t) = \Omega_\xi g(\Omega_{\xi^{-1}} \Omega_\xi x, t) = \Omega_\xi g\mid_{U_i'}(x,t)$$

$$= \Omega_\xi (M_i' x + m_i' t + b_i') = A'(M_i' x' + m_i' t + b_i') + b'$$

So indeed, we see that $h$ is a polygonal map.

2. Now recall the glue lemma:

**Lemma.** *[16, page 108] If two continuous maps $p : A \to X$ and $q : B \to X$ coincide on the overlap on their domain, and $A \cup B = X$, and $A$ and $B$ are either both closed or both open, then*

$$r(x) = \begin{cases} p(x) & x \in A \\ q(x) & x \in B \end{cases}$$

*is a continuous map.*

**Corollary.** *Further, if $p : A \to Im(A)$ and $q : B \to Im(B)$ are homeomorphisms, then $r : X \to X$ is a homeomorphism.*

we have that $f(\bullet, t)$ and $g(\bullet, t)$ are homeomorphisms for all $t \in [0,1]$. Then $\Omega_\nu \circ f(\bullet, t) \circ \Omega_{\nu^{-1}}$ and $\Omega_\xi \circ g(\bullet, t) \circ \Omega_{\xi^{-1}}$ are also therefore homeomorphisms, as affine transformations are homeomorphisms. The domain $A$ in this case is $\mathbb{R}^2 \times [0, 1/2] \times [0,1]$, and $B$ is $\mathbb{R}^2 \times [1/2, 1] \times [0,1]$. As already noted, the functions coincide on $\mathbb{R}^2 \times \{1/2\} \times [0,1]$. We note that both the domains are closed, as the product of closed sets is closed. Therefore, by the corollary noted above, our function $h(\bullet, t)$ is a homeomorphism for any $t \in [0.1]$.

3. We have that $f(\bullet, t)$ and $g(\bullet, t)$ restricted on the set $\mathbb{R}^2 \times \{0,1\}$ is the identity map for all $t \in [0,1]$. Suppose $x \in \mathbb{R}^2 \times \{0,1\}$. Then recalling the definitions of $\Omega_\nu$ and $\Omega_\xi$, we get

$$h(\bullet, t)(x) = h(x,t)$$

$$= \begin{cases} \Omega_\nu f(\Omega_{\nu^{-1}} x, t) = \Omega_\nu f(x,t) = \Omega_\nu x = x & x \in \mathbb{R}^2 \times \{0\} \\ \Omega_\xi g(\Omega_{\xi^{-1}} x, t) = \Omega_\xi g(x,t) = \Omega_\xi x = x & x \in \mathbb{R}^2 \times \{1\} \end{cases}$$

for all $t \in [0,1]$.

4. We have that $f(\bullet, 0)$ and $g(\bullet, 0)$ are identity maps. Then

$$h(\bullet, 0)(x) = h(x, 0)$$

$$= \begin{cases} \Omega_\nu f(\Omega_{\nu^{-1}} x, 0) = \Omega_\nu \Omega_{\nu^{-1}} x = x \\ \Omega_\xi g(\Omega_{\xi^{-1}} x, 0) = \Omega_\xi \Omega_{\xi^{-1}} x = x \end{cases}$$

So $h(\bullet, 0) : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ is the identity map.

So $h$ is an isotopy.

We now proceed to show that $h$ takes $U \circ T$ to $U' \circ T'$. We have

$$h(\bullet, 1)[U \circ T]$$

$$= h(\bullet, 1)[\{(x, y, u) \in \mathbb{R}^2 \times [0, 1] \mid (x, y, 2u - 1) \in T \text{ or } (x, y, 2u) \in U\}]$$
$$= h(\bullet, 1)[\{x \in \mathbb{R}^2 \times [0, 1] \mid \Omega_{\xi^{-1}} x \in T \text{ or } \Omega_{\nu^{-1}} x \in U\}]$$

Recalling our definition of $h$ in equation 2, suppose $x = (x_1, x_2, u) \in \mathbb{R}^2 \times [0, 1]$. Further suppose $\Omega_{\xi^{-1}} x \in T$, i. e. $(x_1, x_2, 2u - 1) \in T$. Then we have that $2u - 1 \in [0, 1]$, so we must have $u \in [1/2, 1]$. So $h(x, 1) = \Omega_\xi g(\Omega_{\xi^{-1}} x, 1)$. Observing that $g(\Omega_{\xi^{-1}} x, 1) \in g(\bullet, 1)[T] = T'$, we have $h(x, 1) \in \Omega_\xi[T']$.

Now instead suppose $\Omega_{\nu^{-1}} x \in U'$, i. e. $(x, y, 2u) \in U'$. Then we have that $2u \in [0, 1]$, so we must have $u \in [0, 1/2]$. So $h(x, 1) = \Omega_\nu f(\Omega_{\nu^{-1}} x, 1)$. Observing that $f(\Omega_{\nu^{-1}} x, 1) \in f(\bullet, 1)[U] = U'$, we have $h(x, 1) \in \Omega_\nu[U']$.

Therefore we have

$$h(\bullet, 1)[U \circ T] \subset U' \circ T' = \{y \in \mathbb{R}^2 \times [0, 1] \mid \Omega_{\xi^{-1}} y \in T' \text{ or } \Omega_{\nu^{-1}} y \in U'\}$$

For the converse inclusion, suppose $y = (y_1, y_2, u) \in \mathbb{R}^2 \times [0, 1]$. We want to express $h(\bullet, 1)(x) = h(x, 1) = y$ for some $x$, such that $\Omega_{\xi^{-1}} x \in T$ or $\Omega_{\nu^{-1}} x \in U$.

Further, suppose $\Omega_{\xi^{-1}} y \in T'$. Then noting from condition 3 of isotopies that $g^{-1}(\bullet, 1)$ is bijective, we have $g^{-1}(\bullet, 1)(\Omega_{\xi^{-1}} y) \in g^{-1}(\bullet, 1)[T'] = T$. Then put $x := \Omega_\xi g^{-1}(\bullet, 1)(\Omega_{\xi^{-1}} y)$. So $\Omega_\xi g(\bullet, 1)\Omega_{\xi^{-1}} x = y$. We have $\Omega_{\xi^{-1}} x \in T$, from which it follows that its third component is in $[1/2, 1]$. Therefore we have

$$h(x, 1) = h(\bullet, 1)(\Omega_\xi g^{-1}(\bullet, 1)(\Omega_{\xi^{-1}} y)) = \Omega_\xi g(\bullet, 1)\Omega_{\xi^{-1}} \Omega_\xi g^{-1}(\bullet, 1)(\Omega_{\xi^{-1}} y) = y$$

Now instead, suppose $\Omega_{\nu^{-1}} y \in U'$. Again, from condition 3 of isotopies that $f^{-1}(\bullet, 1)$ is bijective, we have $f^{-1}(\bullet, 1)(\Omega_{\nu^{-1}} y) \in f^{-1}(\bullet, 1)[U'] = U$. Then put $x := \Omega_\nu f^{-1}(\bullet, 1)(\Omega_{\nu^{-1}} y)$. So $\Omega_\nu f(\bullet, 1)\Omega_{\nu^{-1}} x = y$. We have $\Omega_{\xi^{-1}} x \in U$, from which it follows that its third component is in $[0, 1/2]$. Therefore we have

$$h(x, 1) = h(\bullet, 1)(\Omega_\nu f^{-1}(\bullet, 1)(\Omega_{\nu^{-1}} y) = \Omega_\nu f(\bullet, 1)\Omega_{\nu^{-1}} \Omega_\nu f^{-1}(\bullet, 1)(\Omega_{\nu^{-1}} y = y$$

Therefore we have

$$U' \circ T' \subset h(\bullet, 1)[U \circ T] = h(\bullet, 1)[\{x \in \mathbb{R}^2 \times [0, 1] \mid \Omega_{\xi^{-1}} x \in T \text{ or } \Omega_{\nu^{-1}} x \in U\}]$$

and $h$ has been shown to be an isotopy from $U \circ T$ to $U' \circ T'$. □

We had provided an expository full proof of the above proposition regarding of a property invariant under equivalence of tangles by exhibiting an explicit isotopy and going through each condition to confirm that it is an isotopy and showing that it in fact brings the first tangle to the second. However, in subsequent proofs regarding isotopic tangles, we will a short summary of a possible proof.

## 5.8 Identity Tangle

**Definition 151.** For $n \in \mathbb{N}$, denote $L_n := [(n,0,0),(n,0,1)]$, with orientation $((n,0,0),(n,0,1))$ and $K_n := [(n,0,1),(n,0,0)]$, with orientation $((n,0,1),(n,0,0))$. Given sequence $\Delta = (\delta_1, \cdots, \delta_k) \in \{+,-\}^{\{1,\cdots,k\}}$, the tangle $\mathrm{id}_\Delta$ defined as

$$\mathrm{id}_\Delta := \bigcup_{\Delta(n)=+} \{(L_n, \omega_{L_n})\} \cup \bigcup_{\Delta(n)=-} \{(K_n, \omega_{K_n})\}$$

shall be called the "identity tangle of type $\Delta$".

**Example 152.** Figure 19 gives an example of an identity tangle.

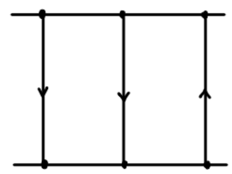

Figure 19: The identity tangle for $\Delta = (+,+,-)$

**Proposition 153.** *If $T$ is a tangle, then $T \circ \mathrm{id}_{\sigma(T)}$ and $\mathrm{id}_{\tau(T)} \circ T$ are isotopic to $T$.*

*Proof.* Writing a full formal proof will be very cumbersome and perhaps even obfuscating, so we merely give a brief sketch. Put $T = \bigcup_{i \in I}\{(L_i, \omega_i)\}$, where $L_i$ is an arc, and $\omega_i = (M_1, \cdots, M_{n(i)})$ is its orientation.

We use the convenient fact that for some $(k,l)$, $\partial(T) = T \cap \mathbb{R}^2 \times \{0,1\} = [k] \times \{(0,0)\} \cup [l] \times \{(0,1)\}$ that we included in our definition of a tangle. We note that this implies that there exists small enough $\varepsilon$ such that $1/2 > \varepsilon > 0$ such that $\bigcup_i \omega_i \cap \mathbb{R}^2 \times [1, 1-\varepsilon] = T \cap \mathbb{R}^2 \times \{1\}$, as in figure 20.

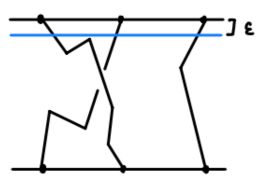


Figure 20: An $\varepsilon$ small enough can always be taken.

The join $T \circ \mathrm{id}_{\sigma(T)}$ therefore has a region $\mathbb{R}^2 \times [1, 1/2 - \varepsilon/2]$ in which all the lines go upward. Since $\varepsilon < 1/2$, we can divide each region nearby into $m$-blocks $[m+1/2, m-1/2] \times \mathbb{R} \times [0,1]$ for each $m \in \mathbb{N}$.

For each $m$-block intersect with the region $\mathbb{R}^2 \times [1, 1/2 - \varepsilon/2]$, therefore, there exists only one line connecting the main tangle to the starting point of the tangle, going in an upward direction. We can, for each region, apply an affine transformation making the lines straight in each region $[m+1/2, m-1/2] \times \mathbb{R} \times [1, 1/2 - \varepsilon/2]$, as in figure 21. Then denote function $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ defined by

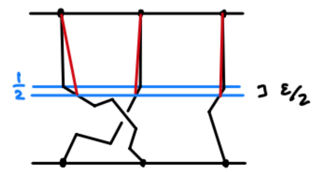


Figure 21: Rectify the portions above $1/2-\varepsilon/2$ into the arcs indicated in red.

$$f(x,y,z)=\begin{cases}(x,y,(2\varepsilon/(1+\varepsilon))z+(1-2\varepsilon/(1+\varepsilon))) & z\in\mathbb{R}^2\times[1/2-\varepsilon/2,1]\\ (x,y,2z) & z\in\mathbb{R}^2\times[0,1/2-\varepsilon/2]\end{cases}$$

Then define $h:(\mathbb{R}^2\times[0,1])\times[0,1]\to(\mathbb{R}^2\times[0,1])\times[0,1]$ by $h(\mathbf{x},t)=t\cdot f(\mathbf{x})+(1-t)\cdot\mathbf{x}$. Then $h$ is in fact an isotopy, which takes $T\circ\mathrm{id}_{\sigma(T)}$ to $T$. Via a similar process we have $\mathrm{id}_{\tau(T)}\circ T$ is isotopic to $T$. □

**Corollary 154.** *For tangle $T$, we have $[T]\circ[\mathrm{id}_{\sigma(T)}]=[\mathrm{id}_{\tau(T)}]\circ[T]=[T]$.*

### 5.9 The Composition of Tangles is Associative

**Proposition 155.** *Given any three composible tangles $[T]$, $[U]$, $[V]$, we have that $([T]\circ[U])\circ[V]=[T]\circ([U]\circ[V])$.*

*Proof.* It suffices to exhibit an isotopy from $(T\circ U)\circ V$ to $T\circ(U\circ V)$. The following map $f:\mathbb{R}^2\times[0,1]\to f:\mathbb{R}^2\times[0,1]$

$$f(x,y,z)=\begin{cases}(x,y,z/2+1/2) & z\in\mathbb{R}^2\times[1,1/2]\\ (x,y,z+1/4) & z\in\mathbb{R}^2\times[1/2,1/4]\\ (x,y,2z) & z\in\mathbb{R}^2\times[1/4,0]\end{cases}$$

and then define $h:(\mathbb{R}^2\times[0,1])\times[0,1]\to(\mathbb{R}^2\times[0,1])\times[0,1]$ by $h(\mathbf{x},t)=t\cdot f(\mathbf{x})+(1-t)\cdot\mathbf{x}$. Then it can be verified, (similarly to how we proved proposition 150) that $h$ is a polygonal isotopy, and that it indeed takes $(T\circ U)\circ V$ to $T\circ(U\circ V)$. This only requires checking so we will not go into the details here. □

## 6 The Category of Tangles and the Category of Braids

### 6.1 The Category Tang of Tangles

We can now give a definition of the category of tangles.

**Definition 156.** Denote $\mathfrak{O}:=\bigcup_{n\in\mathbb{N}}\{+,-\}^n$, that is, it is the set of all sequences, including the empty sequence, $(\varepsilon_1,\cdots,\varepsilon_n)$, where $\varepsilon_i\in\{+,-\}$. Denote $\mathfrak{H}$ as the set of all equivalence classes of tangles. Given $(k,l)\in\mathfrak{O}$, denote $\mathrm{Hom}(s,t)$ as the set of all equivalence classes $[T]$ of tangles $T$ such that $s=\sigma(T)$ and $t=\tau(T)$. Given any two sets $\mathrm{Hom}(s,t)$ and $\mathrm{Hom}(t,u)$, define the map

$$\circ_{s,t,u}:\mathrm{Hom}(s,t)\times\mathrm{Hom}(t,u)\to\mathrm{Hom}(s,u)$$

by associating

$$\circ_{s,t,u} : ([T], [T']) \to [T' \circ T]$$

Denote $\circ$ as the indexed set $(\circ_{s,t,u})_{s,t,u\in\mathfrak{O}}$. Then the four tuple $(\mathfrak{O}, \mathfrak{H}, \mathrm{Hom}, \circ)$ is called the "category of tangles", and is denoted **Tang**.

**Proposition 157.** *The category of tangles is a category.*

*Proof.* We proceed to check the conditions.

1. We see that $\mathrm{Hom}(s,t)$ is a subset of $\mathfrak{H}$, so $\mathrm{Hom} : \mathfrak{O} \times \mathfrak{O} \to \mathcal{P}(\mathfrak{H})$.
   (a) Given any tangle $[T]$, we have that it belongs to $\mathrm{Hom}(\sigma(T), \tau(T))$. So $\mathfrak{H} = \bigcup_{s.t\in\mathfrak{O}} \mathrm{Hom}(s,t)$.
   (b) The image of any two elements in the image of Hom are pairwise disjoint because the if $(\sigma(T), \tau(T)) \neq \sigma(T'), \tau(T')$ then $T$ cannot be equal to $T'$.
2. For $s, t, u \in \mathfrak{O}$, denote

$$M_{s,t,u} := Map(\mathrm{Hom}(s,t) \times \mathrm{Hom}(t,u), \mathrm{Hom}(s,u))$$

   (a) Each function $\circ_{s,t,u}$ is indeed in $M_{s,t,u}$ by definition.
   (b) Given $s \in \mathfrak{O}$ identity morphism is given by $\mathrm{id}_s$ as given in definition 151, and that it satisfies the desired properties is due to corollary 154.
   (c) For all $s, t, u, v \in Ob(\mathbb{A})$, and for all $[T] \in \mathrm{Hom}(s,t)$, $[T'] \in \mathrm{Hom}(t,u)$, and $[T''] \in \mathrm{Hom}(u,v)$, we have $([T] \circ [T']) \circ [T''] = [T] \circ ([T'] \circ [T''])$ due to proposition 155.

$\square$

## 6.2 Tensor Product of Tangles

Kassel [12, page 299] mentions that we can define a tensor product of tangles by putting them side by side. Here in this subsection I propound a rigorous mathematical framework for this definition.

We first need to state what it means to place two tangles side by side, and in what situations we are able to do so.

**Definition 158.** Define $S_n : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ as the function which associates

$$S_n : (x, y, z) \mapsto (x, y, z + n)$$

Then $S_n[T]$ is the tangle which shifts tangle $T$ to right $n$ spaces in the $x$ axis. In particular, each arc $L_i$ is transformed to

$$\left\{(x, y, z) \in \mathbb{R}^2 \times [0,1] \mid (x - n, y, z) \in L_i\right\}$$

Note that for nonzero $n$, $S_n[T]$ is no longer is a tangle.

**Definition 159.** Define $A_{1/3} : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ as the function which associates

$$A_{1/3} : (x, y, z) \mapsto (x, y, z/3 + 1/3)$$

So that $A_{1/3}[T]$ is the original tangle $T$ but squeezed in the interval $[1/3, 2/3]$. In particular, each arc $L_i$ is transformed to

$$\{(x, y, z) \in \mathbb{R}^2 \times [0,1] \mid (x, y, z/3 + 1/3) \in L_i\}$$

An illustration of these definitions at work is given when we illustrate an example of a tensor product of tangles in figure 24.

Now given a non-empty tangle $T$, we affix the following notation

$$a_T := \max\{z \mid (x, y, z) \in L \text{ for some arc } L \text{ in tangle } T\}$$

$$b_T := \min\{z \mid (x, y, z) \in L \text{ for some arc } L \text{ in tangle } T'\}$$

**Lemma 160.** *Given a non-empty tangle $T$ of type $(k, l)$, there exists a tangle $U$ again of type $(k, l)$ such that $1 \leq b_U$ and $a_U \leq max\{k, l\}$, and $T \sim T'$.*

*Proof.* It suffices to give an isotopy that squeezes the middle section of $T$ into some $T'$ that satisfies the conditions. Note that $(a_T - (b_T - 1))$ is non-negative. Set $g$ as the function

$$g : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$$

$$g(x, y, z) = ((x - (b_T - 1))/(a_T - (b_T - 1)), y, z)$$

Then $A_{1/3}[g[T]]$ is a tangle that is squeezed within $[1, \max\{k, l\}] \times \mathbb{R} \times [1/3, 2/3]$. Then connect the ends of the squeezed tangle to $[k] \times \{(0,0)\} \cup [l] \times \{(0,1)\}$, which is the set of ends of the original tangle. This gives the desired tangle. □

**Definition 161.** As a mathematical term, we shall call such a tangle $U$ such that $1 \leq b_U$ and $a_U \leq \max\{k, l\}$ as "well behaved". The empty tangle is defined to be well behaved.

**Example 162.** Figure 22 illustrates well behaved and not well behaved tangles.

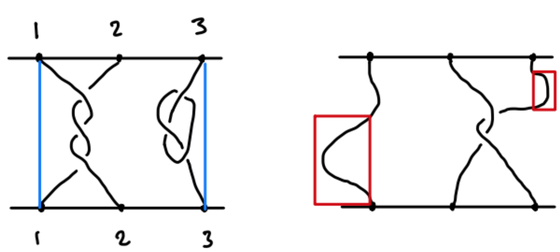


Figure 22: The left tangle is well behaved, while the tangle on the right is not.

**Definition 163.** Given two well behaved tangles $T, T'$ of type $(k, l)$ and $(k', l')$, define

$$N := \max\{k, l\}$$

Then $T$ and $S_N[T']$ do not overlap because they are both well behaved. Put $M_{T,T'}$ as the set $A_{1/3}[T \cup S_N[T']]$. Then extend the upper ends of $M_{T,T'}$ to the set $[l + l'] \times \{(0, 1)\}$, and extend the lower ends of $M_{T,T'}$ to the set $[k + k'] \times \{(0, 0)\}$. Denote this set as $T \otimes T'$, called the "tensor product of $T$ and $T'$".

**Example 164.** The tensor product of two well behaved tangles is illustrated in figures 23, 24, and 25.

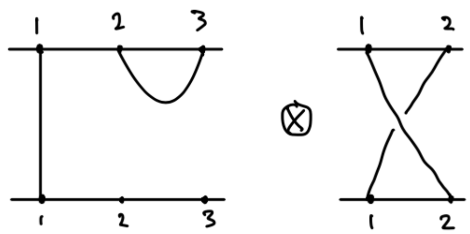


Figure 23: The tensor product of two tangles of type $(3, 1)$ and $(2, 2)$, respectively.

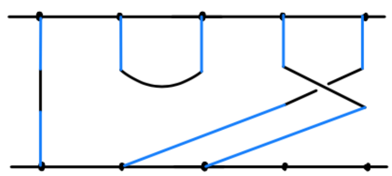

Figure 24: Placing the tangles side by side, and vertically squeezing the tangles and then joining the ends to the top and bottom points.

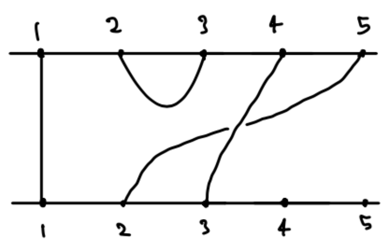


Figure 25: Final result of the tensor product in a more visually appealing illustration. We obtain a tangle of type $(5, 3)$.

**Example 165.** If $T' = \varnothing$, then $T \circ T' = A_{1/3}[T]$. If $T = \varnothing$, then $T \circ T' = A_{1/3}[T']$.

**Proposition 166.** *If $T$ and $T'$ are well behaved tangles, then $T \otimes T'$ is a well behaved tangle.*

*Proof.* Suppose $T, T'$, are two tangles of type $(k, l)$ and $(k', l')$, We need to verify the conditions.

1. The tensor product $T \otimes T'$ is indeed a union of oriented polygonal arcs $(L_i, \omega_i)$, each of the arcs being a subset of $\mathbb{R}^2 \times [0,1]$ by definition.
2. By definition, the ends of the tangles reside in $[k+k'] \times \{(0,0)\} \cup [l+l'] \times \{(0,1)\}$, so $\partial(T) = [k+k'] \times \{(0,0)\} \cup [l+l'] \times \{(0,1)\}$. We have that $\partial(T) = T \cap \mathbb{R}^2 \times \{0,1\}$. So the second condition is also satisfied.

We see that $T \otimes T'$ is a well behaved tangle because the polygonal arcs reside within $[0, \max\{l+l', k+k'\}] \times \mathbb{R} \times [0,1]$. □

**Proposition 167.** *If $T, T', U, U'$ are well behaved tangles, and $T \sim T'$ and $U \sim U'$, then $T \otimes U \sim T' \otimes U'$.*

*That is, the tensor product is invariant under isotopy equivalence of tangles.*

*Proof.* Suppose $T \sim T'$ and $U \sim U'$, where $T, T'$ are of type $(k,l)$, and $U, U'$ are of type $(k', l')$. Put $N := \max\{k,l\}$ and $M = \max\{k', l'\}$. Take $f$ and $g$ as the respective isotopies. Then we can apply $f$ in the region $[1, M] \times \mathbb{R} \times [1/3, 2/3]$ and $g$ in the region $[N+1, N+M]$ □

**Proposition 168.** *Given any tangle $T$, there exists tangle $T'$ such that $T'$ is well behaved and $T \sim T'$.*

*Proof.* We do not go into details, but it is not diffcult to see that we can always give an isotopy that squeezes the sides $T$ to make it well behaved. □

The above proposition allows us to give the following definition.

**Definition 169.** Given two isotopy classes of tangles $[T], [T']$, take $U \in [T]$ and $[U'] \in [T']$ such that $U$ and $T$ are well behaved. Define the tensor product of $[T], [T']$ as

$$[T] \otimes [T'] := [U \otimes U']$$

**Proposition 170.** *Given any three tanlges $T, U, V$, we have $(T \otimes U) \otimes V$ is isotopic to $T \otimes (U \otimes V)$.*

The above proposition is more or less visually clear when we look at how we have constructed the tensor product, so we will omit the proof.

**Corollary 171.** *Given any three tanlges $T, U, V$, we have $([T] \otimes [U]) \otimes [V] = [T] \otimes ([U] \otimes [V])$. That is, the tensor product of three tangles is associative.*

## 6.3 The Tensor Category of Tangles

We now define a functor from **Tang** $\times$ **Tang** to **Tang**.

**Definition 172.** Denote $\otimes$ as the association which takes

$$(\varepsilon_1, \cdots, \varepsilon_n), (\eta_1, \cdots, \eta_m) \in \bigcup_{n \in \mathbb{N}} \{+,-\}^n$$

and maps it to

$$\otimes : ((\varepsilon_1, \cdots, \varepsilon_n), (\eta_1, \cdots, \eta_m)) \longmapsto (\varepsilon_1, \cdots, \varepsilon_n, \eta_1, \cdots, \eta_m)$$

and for any two morphisms $[T] : (\varepsilon_1, \cdots, \varepsilon_n) \to (\eta_1, \cdots, \eta_m)$ and $[T'] : (\varepsilon'_1, \cdots, \varepsilon'_{n'}) \to (\eta'_1, \cdots, \eta'_{m'})$, define

$$\otimes : ([T], [T']) \longmapsto [T] \otimes [T']$$

*Remark* 173. We note the important property that the concatenation of three sequences $\varepsilon, \eta, \delta$ is associative, that is, $(\varepsilon \otimes \eta) \otimes \delta = \varepsilon \otimes (\eta \otimes \delta)$.

**Proposition 174.** *If $\varepsilon, \eta$ are two sequences of $\{+,-\}$, and $\mathrm{id}_\varepsilon, \mathrm{id}_\eta$ are their two identity tangles, then $\mathrm{id}_\varepsilon \otimes \mathrm{id}_\eta$ is the identity tangle of the concatenation $\varepsilon \otimes \eta$.*

*Proof.* Since $\mathrm{id}_\varepsilon \otimes \mathrm{id}_\eta$ is simply $\mathrm{id}_\varepsilon$ and $\mathrm{id}_\eta$ placed side by side, we see that it gives the identity tangle of type $\varepsilon \otimes \eta$. □

**Proposition 175.** *Denote* **Tang** *as the category of all tangles. Denote $\otimes$ as the map which takes two morphisms $[T], [U]$, which are equivalence classes of tangles, and maps them to their tensor product $[T] \otimes [U]$. Denote $I$ as the empty sequence. Denote $\alpha$ as the map which takes any $(\varepsilon, \eta, \delta) \in Ob(\mathbf{Tang}) \times Ob(\mathbf{Tang}) \times Ob(\mathbf{Tang})$ to the identity tangle $\mathrm{id}_{\varepsilon,\eta,\delta} : (\varepsilon \otimes \eta) \otimes \delta \to \varepsilon \otimes (\eta \otimes \delta)$, where we note that $(\varepsilon \otimes \eta) \otimes \delta = \varepsilon \otimes (\eta \otimes \delta)$, as per remark 173. Noting that $I \otimes \varepsilon = \varepsilon \otimes I = \varepsilon$, denote $l_\varepsilon : I \otimes \varepsilon \to \varepsilon$ and $r_\varepsilon : \varepsilon \otimes I \to \varepsilon$ as the identity tangle of type $\varepsilon$.*

*Then $(\mathbf{Tang}, \otimes, I, \alpha, l, r)$, is a tensor category.*

*Proof.* Recall definition 71 that we need to verify the following:

$(\mathbf{Tang}, \otimes, R, \alpha, l, r)$, is a tensor category iff

1. $(\mathbf{Tang}, \otimes)$ is a pre-monoidal category.
2. $I$ is an object of **Tang**.
3. $\alpha$ is an associativity constraint on $(\mathbf{Tang}, \otimes)$.
4. The pentagon axiom is satisfied.
5. $l$ is a left unit constraint.
6. $r$ is a right unit constraint.
7. The triangle axiom is satisfied.

That $(\mathbf{Tang}, \otimes)$ is a pre-monoidal category is to say that $\otimes$ is a functor from $\mathbf{Tang} \times \mathbf{Tang}$ to **Tang**. Indeed, we have that it takes objects to objects and morphisms to morphisms. We need to show that it respects identity and composition.

Suppose $(\mathrm{id}_\varepsilon, \mathrm{id}_\eta)$ is a identity morphism in $\mathbf{Tang} \times \mathbf{Tang}$, where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ and $\eta = (\eta_1, \cdots, \eta_m)$. Then putting the two tangles side by side we have that $\mathrm{id}_\varepsilon \otimes \mathrm{id}_\eta$ gives the identity tangle of type $(\varepsilon_1, \cdots, \varepsilon_n, \eta_1, \cdots, \eta_m) = (\varepsilon_1, \cdots, \varepsilon_n) \otimes (\eta_1, \cdots, \eta_m)$. So identity is respected.

Suppose $([T], [T'])$ and $([U], [U'])$ are two composible morphisms in $\mathbf{Tang} \times \mathbf{Tang}$. Then their composition being $([U], [U']) \circ ([T], [T']) = ([U] \circ [T], [U'] \circ [T'])$, where $[T]$ is stacked on top of $[U]$, and $[T']$ is stacked on top of $[U']$. Then we simply put these two side by side to obtain $([U] \circ [T]) \otimes ([U'] \circ [T'])$. On the other hand, placing $[U]$ next to $[T]$, and $[U']$ next to $[T']$ gives $[T] \otimes [T']$ and $[T'] \otimes [U']$ respectively. Placing $[T] \otimes [T']$ on top of $[U] \otimes [U']$ gives $([U] \otimes [U']) \circ ([T] \otimes [T'])$. Seeing that the two tangles $([U] \circ [T]) \otimes ([U'] \circ [T'])$ and $([U] \otimes [U']) \circ ([T] \otimes [T'])$ are the same, we have that $\otimes$ respects composition. Figure 26 illustrates this. So condition 1 is satisfied.

We have that the empty sequence $I$ is an object of **Tang**. So 2 is satisfied.

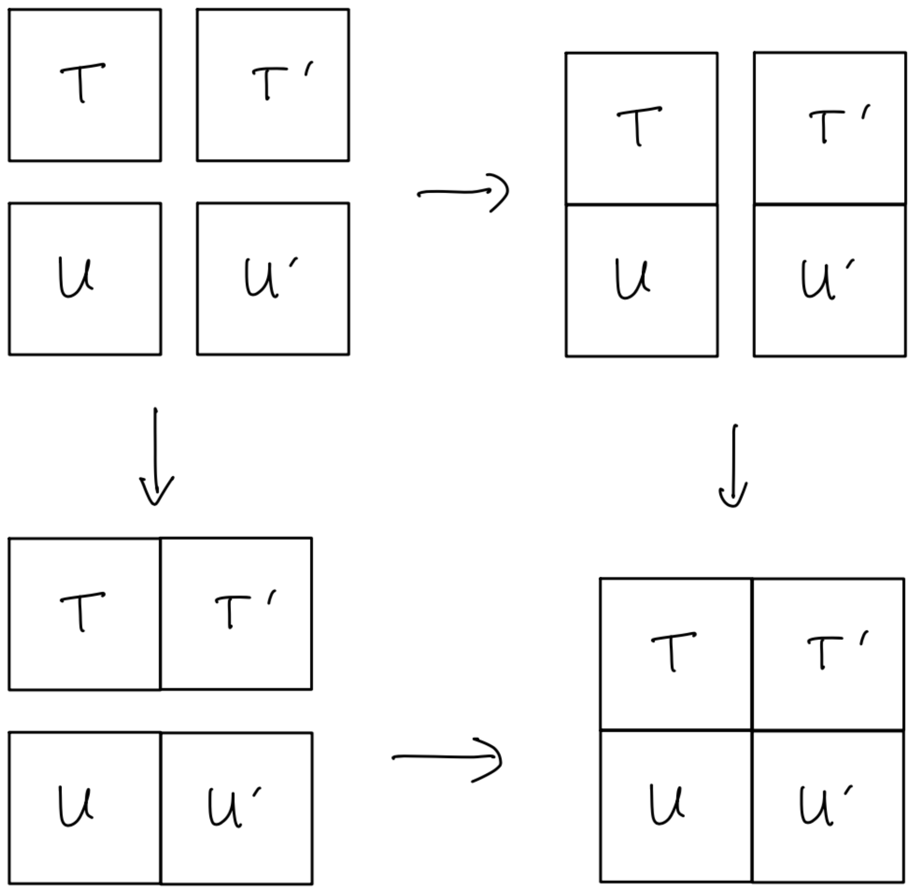


Figure 26: Let us represent tangles as some boxed area in space, where $T, T', U, T'$ are tangles. It does not matter if we compose the tangles first and then apply the tensor product or vice versa. That is, the above illustration commutes.

To say that $\alpha$ is an associativity constraint is to say that for all sequences $\varepsilon, \eta, \delta, \varepsilon', \eta', \delta' \in Ob(\mathbf{Tang})$, the diagram

$$\begin{tikzcd}
(\varepsilon \otimes \eta) \otimes \delta \arrow[r, "\alpha(\varepsilon,\eta,\delta)"] \arrow[d, "(T\otimes U)\otimes V"] & \varepsilon \otimes (\eta \otimes \delta) \arrow[d, "T\otimes(U\otimes V)"] \\
(\varepsilon' \otimes \eta') \otimes \delta' \arrow[r, "\alpha(\varepsilon',\eta',\delta')"'] & \varepsilon' \otimes (\eta' \otimes \delta')
\end{tikzcd}$$

commutes. We recall that $\alpha(\varepsilon, \eta, \delta)$ is the identity tangle of $(\varepsilon \otimes \eta) \otimes \delta$. Note that by corollary 171, we have $([T] \otimes [U]) \otimes [V] = [T] \otimes ([U] \otimes [V])$, and thus

$$\mathrm{id} \circ ([T] \otimes [U]) \otimes [V] = ([T] \otimes [U]) \otimes [V] = [T] \otimes ([U] \otimes [V]) = ([T] \otimes [U]) \otimes [V] \circ \mathrm{id}$$

which proves 3.

The pentagon axiom is the condition that for all objects $\varepsilon, \eta, \delta, \gamma \in Ob(\mathbb{A})$, the following diagram of objects in **Tang** and morphisms in **Tang**

$$\begin{tikzcd}
((\varepsilon\otimes\eta)\otimes\delta)\otimes\gamma \arrow[rr, "\alpha(\varepsilon,\eta,\delta)\otimes\mathrm{id}_\gamma"'] \arrow[d, "\alpha(\varepsilon\otimes\eta,\delta,\gamma)"'] & & (\varepsilon\otimes(\eta\otimes\delta))\otimes\gamma \arrow[d, "\alpha(\varepsilon,\eta\otimes\delta,\gamma)"] \\
(\varepsilon\otimes\eta)\otimes(\delta\otimes\gamma) \arrow[dr, "\alpha(\varepsilon,\eta,\delta\otimes\gamma)"'] & & \varepsilon\otimes((\eta\otimes\delta)\otimes\gamma) \arrow[dl, "\mathrm{id}_\varepsilon\otimes\alpha(\eta,\delta,\gamma)"] \\
& \varepsilon\otimes(\eta\otimes(\delta\otimes\gamma)) &
\end{tikzcd}$$

commutes. As per remark 173, we have that

$$((\varepsilon\otimes\eta)\otimes\delta)\otimes\gamma = (\varepsilon\otimes(\eta\otimes\delta))\otimes\gamma = \varepsilon\otimes((\eta\otimes\delta)\otimes\gamma) = \varepsilon\otimes(\eta\otimes(\delta\otimes\gamma)) = (\varepsilon\otimes\eta)\otimes(\delta\otimes\gamma)$$

and we also have that $\alpha(\varepsilon,\eta,\delta)$ is the identity tangle of $\varepsilon\otimes\eta\otimes\delta$. By definition, $\mathrm{id}_\gamma$ is the identity tangle of $\gamma$. We have that by proposition 174:

$$\alpha(\varepsilon,\eta,\delta)\otimes\mathrm{id}_\gamma = \mathrm{id}_{(\varepsilon\otimes\eta\otimes\delta)\otimes\gamma}$$

Similarly, we also have that $\mathrm{id}_\varepsilon\otimes\alpha(\eta,\delta,\gamma) = \mathrm{id}_{\varepsilon\otimes(\eta\otimes\delta\otimes\gamma)}$.

We also observe that $\alpha(\varepsilon\otimes\eta,\delta,\gamma)$ is the identity tangle of $(\varepsilon\otimes\eta)\otimes\delta\otimes\gamma$, and $\alpha(\varepsilon,\eta,\delta\otimes\gamma)$ is the identity tangle of $\varepsilon\otimes\eta\otimes(\delta\otimes\gamma)$ and $\alpha(\varepsilon,\eta\otimes\delta,\gamma)$ is the identity tangle of $\varepsilon\otimes(\eta\otimes\delta)\otimes\gamma$.

So the diagram reduces to only identity morphisms

$$\begin{tikzcd}
\varepsilon\otimes\eta\otimes\delta\otimes\gamma \arrow[rr, "\mathrm{id}_{\varepsilon\otimes\eta\otimes\delta\otimes\gamma}"'] \arrow[d, "\mathrm{id}_{\varepsilon\otimes\eta\otimes\delta\otimes\gamma}"'] & & \varepsilon\otimes\eta\otimes\delta\otimes\gamma \arrow[d, "\mathrm{id}_{\varepsilon\otimes\eta\otimes\delta\otimes\gamma}"] \\
\varepsilon\otimes\eta\otimes\delta\otimes\gamma \arrow[dr, "\mathrm{id}_{\varepsilon\otimes\eta\otimes\delta\otimes\gamma}"'] & & \varepsilon\otimes\eta\otimes\delta\otimes\gamma \arrow[dl, "\mathrm{id}_{\varepsilon\otimes\eta\otimes\delta\otimes\gamma}"] \\
& \varepsilon\otimes\eta\otimes\delta\otimes\gamma &
\end{tikzcd}$$

which obviously commutes. So 4 is satisfied.

Recall that to say that $l, r$ are left and right unit constraints, respectively, is to say that for all $R$-module homomorphisms $[T]:\varepsilon\to\eta$ the diagrams

$$\begin{tikzcd}
I\otimes\varepsilon \arrow[r, "l(\varepsilon)"'] \arrow[d, "\mathrm{id}_I\otimes[T]"'] & \varepsilon \arrow[d, "{[T]}"] \\
I\otimes\eta \arrow[r, "l(\eta)"'] & \eta
\end{tikzcd}
\qquad
\begin{tikzcd}
\varepsilon\otimes I \arrow[r, "r(\varepsilon)"'] \arrow[d, "{[T]\otimes\mathrm{id}_I}"'] & \varepsilon \arrow[d, "{[T]}"] \\
\eta\otimes I \arrow[r, "r(\eta)"'] & \eta
\end{tikzcd}$$

commute. Indeed, since $I\otimes\varepsilon = \varepsilon\otimes I = I$, and $r(\varepsilon) = l(\varepsilon) = \mathrm{id}_\varepsilon$, and $r(\eta) = l(\eta) = \mathrm{id}_\eta$, and $\mathrm{id}_I\otimes[T] = [T]\otimes\mathrm{id}_I = [T]$, the diagrams become

$$\begin{tikzcd}
\varepsilon \arrow[r, "\mathrm{id}_\varepsilon"'] \arrow[d, "{[T]}"'] & \varepsilon \arrow[d, "{[T]}"] \\
\eta \arrow[r, "\mathrm{id}_\eta"'] & \eta
\end{tikzcd}
\qquad
\begin{tikzcd}
\varepsilon \arrow[r, "\mathrm{id}_\varepsilon"'] \arrow[d, "{[T]}"'] & \varepsilon \arrow[d, "{[T]}"] \\
\eta \arrow[r, "\mathrm{id}_\eta"'] & \eta
\end{tikzcd}$$

which again, clearly commutes. We see that identity is always an isomorphism. So 5 and 6 are satisfied.

Recall that here, the triangle axiom is satisfied iff for all sequences $\varepsilon, \eta$, the diagram

$$\begin{tikzcd}
(\varepsilon \otimes I) \otimes \eta \arrow[rr, "\alpha(\varepsilon, I, \eta)"] \arrow[dr, "r_\varepsilon \otimes \mathrm{id}_\eta"'] & & \varepsilon \otimes (I \otimes \eta) \arrow[dl, "\mathrm{id}_\varepsilon \otimes l_\eta"] \\
& \varepsilon \otimes \eta &
\end{tikzcd}$$

commutes. Again, this reduces to a diagram of only identity morphisms. So condition 7 is satisfied.

Therefore $(\mathbf{Tang}, \otimes, R, \alpha, l, r)$, is a tensor category. $\square$

## 6.4 Projection onto a 2D Plane

Denote $\pi_1$ as the map which takes

$$\mathbb{R}^3 \to \mathbb{R}^2$$

$$(x, y, z) \mapsto (x, z)$$

Given a tangle, $T = \bigcup_i \{(L_i, \omega_i)\}$, we can consider its projection onto the plane. However $\pi_1[T]$ collapses the tangles to 2 dimensions does not contain enough information about when a given an intersection of two tangles, which one crosses over the other.

**Definition 176.** [12, page 246] Given any tangle $T$, and projection $\pi_1[T]$, say that a point $x$ is a "crossing point" of two arcs $\pi_1[L_i]$, $\pi_1[L_j]$ iff $x \in \pi_1[L_i] \cap \pi_1[L_j]$. A point is said to be a "crossing point of $T$" iff there exist two polygonal arcs of $T$ such that it is a crossing point of the two arcs.

**Definition 177.** [12, page 246] Given crossing point $x$, the unique natural number $n$ that corresponds to the total number of unique arcs $L_i$ such that $x \in L_i$ is said to be the "order of $x$".

**Example 178.** In figure 27, we give an illustration of a tangle with exactly one crossing point of order 2.

**Definition.** [12, page 246] The projection $\pi_1[T]$ of a tangle $T$ is said to be "regular" iff all its crossing points are of order 2.

**Proposition 179.** *Given any tangle $T$, there exists an isotopic tangle $U$ such that any crossing point of $\pi_1[U]$ has order 2.*

*Proof.* Let $U$ be any tangle isotopic to $T$. Then the number of crossing points of $\pi_1[U]$ is finite.

For polygonal arc $L_i$ of $T$, note that we required that two arcs cannot intersect one another. Given any two consecutive vertices $(p, q)$ on a polygonal arc $L_i$ and likewise $(p', q')$ of $L_j$, we have that

$$[p, q] \cap [p', q'] = \varnothing$$

and there exists some $\delta(p, q, p', q') > 0$ such that any two points $x \in [p, q]$ and $y \in [p', q']$ have at least distance $\delta(p, q, p', q')$ between them. Then a perturbation of at most $\delta(p, q, p', q')$ of the vertices $p, q, p'$ or $q'$ cannot result in an intersection of two arcs.

Furthermore, given the set of all crossing points $\{x_i\}_i$ there exists a nonzero minimum distance between any two crossing points. Denote it as $m$.

Since the set of all pairs of consecutive vertices of arcs in $T$ are finite, the set

$$\{\frac{1}{2}\delta(p, q, p', q')\}_{p,q,p',q'} \cup \{m\}$$

is also finite. Denote $\mu$ as the least element of this set. Then moving any vertex a distance of $\mu$ does not change the tangle under isotopy.

Now notice that if we perturb a vertex by a distance of $\frac{1}{2}\mu$, isotopy is respected and we do not result in a crossing of two arcs.

Number each crossing point of larger than order 2 and denote it as $x_i$, where $i \in \mathbb{N}$, and again for each $x_i$, number each arc that crosses it as $L_{(i,j)}$, where $j \in \mathbb{N}$. Given an arc, we can assume that it has more than two vertices because we can add one on a straight line. We have that there are two vertices of $L_{(i,j)}$ that form a straight line segment that crosses $x_i$, at least one which is not the beginning or end of the arc. Take either one of the two vertices and denote it as $v_{(i,j)}$.

Then the set of all $(i, j)$ is finite. Denote $n$ as the cardinality of this set. In this case, we will say that "$T$ has total redundant crossing number $n$". We desire to show that any $T$ is isotopic to some $U$ such that $U$ has total redundant crossing number 0.

For tangle $T$ of total redundant crossing number $n$, denote $A := L_{(i,j)}$ and $V := v_{(i,j)}$ for some pair $(i, j)$.

Perturb the arc by alterning its $V_m = (V_1^m, V_2^m, V_3^m)$ vertex by redefining it as

$$V_m := (V_1^m + \frac{1}{2}\mu, V_2^m, V_3^m)$$

if the line is not horizontal, or

$$V_m := (V_1^m, V_2^m, V_3^m \pm \frac{1}{2}\mu)$$

if the line is horizontal. The sign is $+$ if we have $V_3^m - \frac{1}{2}\mu < 0$ and $-$ if we have $V_3^m + \frac{1}{2}\mu > 1$. We observe that this movement of a vertex at most moves any point in tangle $T$ by $\frac{1}{2}\mu$. Then we see that this movement cannot result in passing an arc over another because of the conditions on $\mu$, and for the same reason, it cannot increase the crossing number of some other crossing point. Further, the crossing at $x_i$ is of order one less than it was.

Therefore we obtained a tangle of total crossing number $n - 1$, isotopic to $T$. Repeating this process $n$ times obtains a tangle isotopic to $T$ of redundant crossing number 0. □

So given any tangle $T$, we are able to represent it in a two dimensional plane.

**Definition 180.** [12, page 260] We say that $L$ is a tangle diagram iff it is a regular projection of some tangle $T$, along with the information at each crossing point indicating which arc is above the other.

When trying to describe a tangle, it is therefore enough to draw it in two dimensions as a regular projection, where we indicate for each crossing, which line is on top and which line is on the bottom. Indeed, the examples that we have provided of tangles on the two dimensional presentation of this dissertation are examples of regular projections of tangles. We provide a non-regular projection in the following example.

**Example 181.** The tangle projection in figure 27 is not regular.

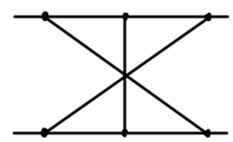

Figure 27: Example of a tangle projection of a tangle with one crossing point of order two. The tangle projection is not regular.

**Definition 182.** Two tangle diagrams $\pi_1[T]$ and $\pi_1[U]$ will be called "isotopic" iff $T$ and $U$ are isotopic.

**Definition 183.** A map $f : (\mathbb{R} \times [0,1]) \times [0,1] \to (\mathbb{R} \times [0,1])$ is said to be a "planar isotopy" iff

1. It is a piecewise linear map.
2. $f(\bullet, t)$ is a homeomorphism for all $t \in [0,1]$.
3. $f(\bullet, t)$ restricted on the set $(\mathbb{R} \times \{0,1\})$ is the identity map for all $t \in [0,1]$.
4. $f(\bullet, 0) : \mathbb{R}^2 \times [0,1] \to \mathbb{R} \times [0,1]$ is the identity map.

**Definition 184.** We will say that two tangle diagrams $\pi_1[T]$ and $\pi_1[U]$ are planar isotopic iff there exists polygonal isotopy $f$ such that $f(\bullet, 1)[\pi_1[T]] = \pi_1[U]$.

We remark the following fact, but will not prove it.

**Proposition 185.** *If two tangle diagrams $\pi_1[T]$ and $\pi_1[U]$ are planar isotopic, then they are isotopic.*

## 6.5 Braids

**Definition 186.** [12, page 262] A tangle $T$ is called a "braid" iff:

1. $\sigma(T)$ and $\tau(T)$ are sequences of only $+$ signs.
2. There are no closed loops in $T$.
3. Any polygonal arc in $T$ only goes downward, that is, if $p = (x, y, z)$ and $q = (x', y', z')$ are two vertices of arc $L$, and $q$ comes after $p$ sequentially, then $z' < z$.

*Remark* 187. Note that this means that $T$ must be of type $(n, n)$ for some $n$.

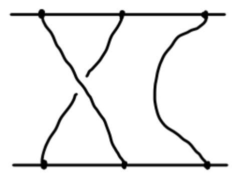

Figure 28: Example of a braid.

We will denote the sequence $(+, \cdots, +)$ of $n$ "+" signs as $[[n]]$.

A tangle diagram of a braid will be called a braid diagram. Two braid diagrams $\pi_1[T]$ and $\pi_1[U]$ will be called "isotopic" iff $T$ and $U$ are isotopic.

**Example 188.** An example of a braid is given in figure 28.

**Example 189.** Two counterexamples of a braid is given in figure 29.

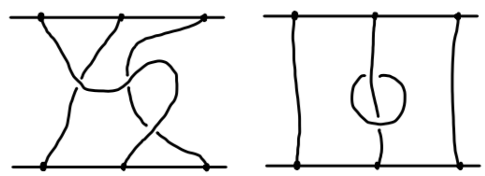

Figure 29: Two counterexamples of a braid.

**Definition 190.** [12, page 264] The tangle underswap is the braid that represents swaps $i$ with $i + 1$, where point $i$ goes underneath and point $i + 1$ goes above. This is represented by figure 30. It is denoted as $X^i_+$.

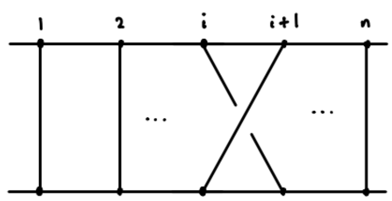


Figure 30: The braid underswap $X^i_+$

**Definition 191.** [12, page 264] The tangle overswap is the braid that represents swaps $i$ with $i + 1$, where point $i + 1$ goes underneath and point $i$ goes above. This is represented by figure 31. It is denoted as $X^i_-$.

**Proposition 192.** *Given any braid $T$ with regular projection, there exists braid $U$ with regular projection such that $T \sim U$ and each crossing of $\pi_1[U]$ is aligned either above or below one another with respect to the last axis, which goes from 0 to 1.*

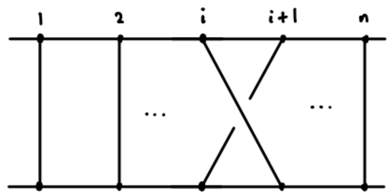


Figure 31: The braid overswap $X^i_-$.

*Proof.* Given any braid $T$, there exists $\varepsilon' > 0$ such that any two crossings and any two vertices, when perturbed, do not change the tangle under isotopy. So for any two crossings on the projected $T$ braid such that they sit at the same height in the last axis, we are able to move one of the crossings some appropriate $\varepsilon < \varepsilon'$ up or down. Repeat until there are no more crossings that sit at the same height.

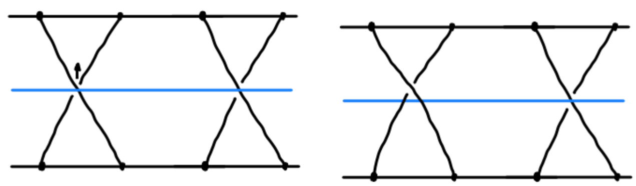

Figure 32: Moving a crossing upward.

Then the projection of this perturbed tangle satisfies the required conditions. □

**Definition 193.** We will call a braid $T$ a "well-drawn braid" iff it is expressed as a composition of a (possibly empty) sequence of underswaps and overswaps. We will call a braid diagram "well drawn" iff it is the projection of some well drawn braid. I propound that we call such a sequence a "swap decomposition of $T$". In particular, when there are no crossings, as defined in the identity tangle (identity braid), then the sequence is empty.

**Corollary 194.** *Given any braid $T$, there exists $U$ such that $T \sim U$ and $U$ well-drawn.*

*Proof.* Taking $U$ as in the above proposition and its projection $\pi_1[U]$, for each crossing sequentially occuring from top to bottom, compose the appropriate undercrossings or overcrossings in the same sequential order to obtain the desired isotopic tangle. We omit the details of the isotopy. □

Then for any braid class $[T] \in \mathrm{Hom}(\mathbf{Braid})$ there exists sequence $(B_1, \cdots, B_n)$ of braids, such that $[T] = [B_1 \circ \cdots \circ B_n]$, where $B_i$ are underswaps or overswaps. I hence propound that we call $(B_1, \cdots, B_n)$ a "swap decomposition of $[T]$".

**Example 195.** Figure 33 illustrates a swap decomposition of a braid.

## 6.6 Reidemeister Moves

When we project polygonal arcs, or in our specific case, braids, onto a 2 dimensional plane, we would like to have a formal way to give a notion of isotopy of a braid diagram. The question

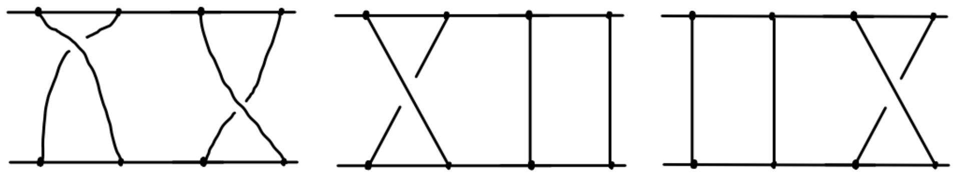

Figure 33: The first braid is the second braid composed with the third.

is, if we have two braids $T$ and $U$, such that their projections are regular, then how can we determine whether they are isotopic just by looking at the tangle diagrams $\pi_1[T]$ and $\pi_1[U]$?

A solution is to present possible ways to locally manipulate a tangle diagram such that isotopy is preserved, which have been given by Riedemeister [19]. In knot theory, there are formally four Reidemeister moves (move 0, move 1, move 2, and move 3), but our focus will be on move 2 and move 3, illustrated in [12, page 248].

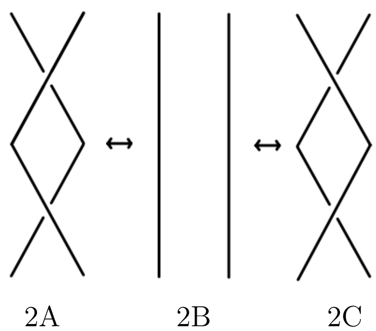


Figure 34: Reidemeister Move 2.

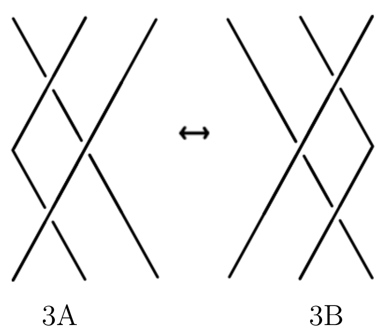


Figure 35: Reidemeister move 3.

We will say that 2A, 2B and 2C are equivalent braid diagrams. We will similarly say that 3A and 3B are equivalent braid diagrams [12, page 263]. [12, page 248] informally describes an application of a Reidemeister move on a braid diagram $\pi_1[T]$ as a graphical replacement of a portion of the diagram with an equivalent braid diagram; a 2 dimensional transformation. However, I here have developed the theory of braids such that we can express them as underswaps and overswaps. I therefore propound the following precise definition as

Reidemeister moves on well-drawn (see definition 193) braids.

**Definition 196.** Given a well-drawn braid $T$ and a swap decomposition $(B_1, \cdots, B_n)$ of $T$, the Reidemeister transformation between $2A$ and $2B$ corresponds to the operation

$$(B_1, \cdots, B_j, X^i_+, X^i_-, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, B_{j+3}, \cdots, B_n) \tag{3}$$

which removes or adds $(X^{i+1}_+, X^{i+1}_-)$ from the sequence. The Reidemeister transformation between $2B$ and $2C$ corresponds to the operation

$$(B_1, \cdots, B_j, X^i_-, X^i_+, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, B_{j+3}, \cdots, B_n) \tag{4}$$

which removes or adds $(X^{i+1}_-, X^{i+1}_+)$ from the sequence. The Reidemeister transformation between $3A$ and $3B$ corresponds to the operation

$$(B_1, \cdots, B_j, X^i_+, X^{i+1}_+, X^i_+, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, X^{i+1}_+, X^i_+, X^{i+1}_+, B_{j+3}, \cdots, B_n) \tag{5}$$

We also need a move to represent isotopy.

**Definition 197.** Given a well-drawn braid $T$ and a swap decomposition $(B_1, \cdots, B_n)$ of $T$, the planar isotopy transformation corresponds to

$$(B_1, \cdots, B_j, X^i_\pm, X^j_\pm, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, X^i_\pm, X^j_\pm B_{j+3}, \cdots, B_n)$$

for any $i, j$ such that $i \neq j+1$ or $j \neq i+1$.

**Example 198.** For example, an application of move 2 would be as given in figure 36.

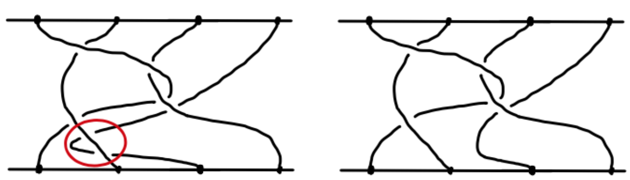

Figure 36: Application of Reidemeister move 2 on the portion circled in red. We obtain the illustration on the right.

The local alteration of a well drawn braid diagram under Reidemeister moves or planar isotopy transformations clearly obtains isotopic braids. On the contrary, in fact the Reidemeister moves 2 and 3 and planar isotopy transformations are enough for two well drawn braid diagrams to be isotopic. This is made precise in the following proposition from which it follows a more general corollary, which is merely mentioned as a fact in [12, page 263].

**Proposition 199.** *If $U$ and $T$ are well-drawn braids, then they are isotopic iff $U$ is obtained from $T$ from successive operations of (3), (4), and (5) in definition 196 and planar isotopy transformations.*

We present merely a brief sketch of a proof, inspired by [20, page 18, 19].

*Proof.* We recall proposition 135, which states that isotopies of tangles are obtained from $\Delta$=moves. A valid $\Delta$-move in braids is one in which the application of a $\Delta$-move results in a braid, that is, we cannot have a $\Delta$-move that makes an arc go upward in some portion of the diagram, as in figure 37. Then any large $\Delta$-move can be given as a composition of one of the following smaller delta moves which does the following:

1. Crosses over a new single crossing with no vertex.

2. Crosses over a new single arc segment with no vertex.

3. Crosses over two new arc segments joined by a vertex.

4. Crosses over nothing new.

Then it can be shown that each of these moves corresponds to some combinations of Reidemeister moves and planar isotopy moves. For example, number 3 corresponds to applying a $\Delta$-move from A to C in figure 37. C is isotopic to D. So we see that this is simply the application of Reidemeister move 2 to A.

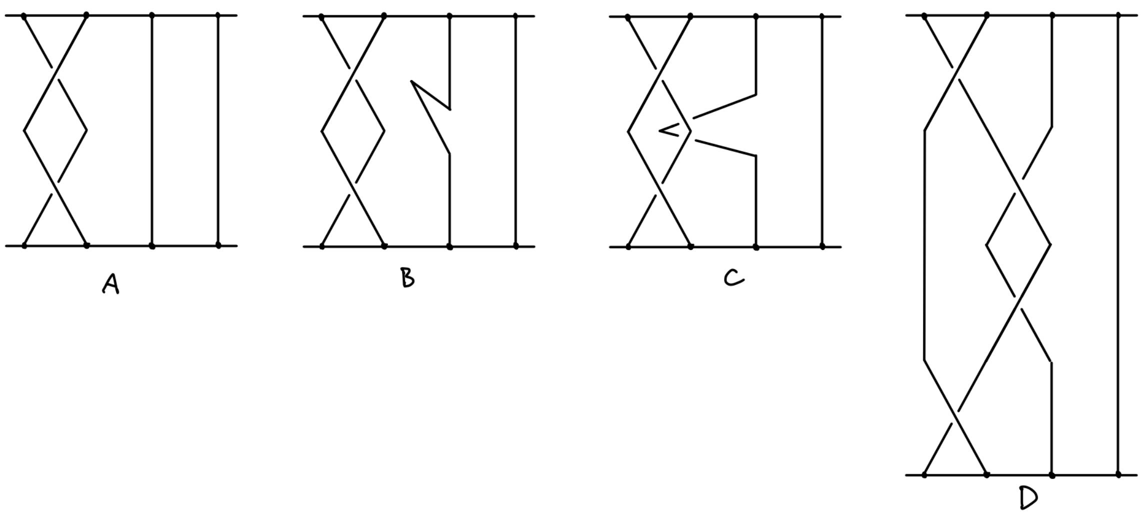


Figure 37: Some braids. The $\Delta$-move from A to B is invalid, because B is not a braid, but the $\Delta$-move from A to C is valid. The projection of C is planar isotopic to the projection of D.

$\square$

**Corollary 200.** *[12, page 263] If $T$ and $U$ are braids, and $\pi_1[U]$ is obtained from $\pi_1[T]$ by successive applications of Reidemeister moves, along with a homeomorphic 2 dimensional polygonal transformation, then $T \sim U$.*

Our discussion leads naturally to the Artin braid group of $n$ points. Braids are expressible as elements of a group. From our discussion, it is then easy to find a group presentation of the braid group. We will not develop this theory here.

### 6.7 The Submonoidal Category of Braids

**Proposition 201.** *Define $Ob(S)$ as the set of all "+" sequences of any length. A morphism from a sequence length $m$ to length $n$ is empty if $m \neq n$, and is the set of all equivalence classes of braids type $(n, n)$ if $m = n$. The composition is defined as the tangle composition.*

*Then this defines a subcategory of* **Tang**.

*Proof.* Looking at the conditions noted in definition 59, it is immediately seen that all of them are true by definition. □

We denote the braid category as **Braid**.

Recall how we defined the submonoidal category in definition 73.

**Definition 202.** For the tensor category $(\mathbf{Tang}, \otimes, I, \alpha, l, r)$, and the braid subcategory **Braid** of **Tang**, define:

1. $\otimes_{\mathbf{B}}$ as the restriction of the functor of $\boxtimes$ on the objects and morphisms **Braid**.

2. $I_{\mathbf{B}} := I$.

3. $\alpha_{\mathbf{B}}$ as the restriction of $\alpha$ on objects of **Braid**.

4. $l_{\mathbf{B}}$ and $r_{\mathbf{B}}$ as the restriction of $l$ and $r$ on the objects and morphisms of **Braid**.

**Proposition 203.** $(\mathbf{Braid}, \otimes_{\mathbf{B}}, I_{\mathbf{B}}, \alpha_{\mathbf{B}}, l_{\mathbf{B}}, r_{\mathbf{B}})$ *is a tensor category.*

*Proof.* We appeal to proposition 74, and verify the required conditions.

1. For all composible $f, g \in \mathbf{Braid}$, we have $g \boxtimes f \in \mathbf{Braids}$. This is true because placing two braids side by side is another braid.

2. $I \in \mathbb{B}$. The empty sequence is an object of **Braid**.

3. For all $\varepsilon, \eta, \delta \in Ob(\mathbb{B})$, we have $\alpha(\varepsilon, \eta, \delta) \in \mathrm{Hom}(\mathbb{B})$, for indeed, the concatenation of finite sequences of "+" symbols is a finites sequence of "+" symbols.

4. For all $\varepsilon \in Ob(\mathbb{B})$, we have $l(\varepsilon), r(\varepsilon) \in \mathrm{Hom}(\mathbb{B})$. This is true, because by definition, it is the identity tangle of type $\varepsilon$, which is also a braid.

So $(\mathbf{Braid}, \boxtimes_{\mathbf{B}}, I_{\mathbf{B}}, \alpha_{\mathbf{B}}, l_{\mathbf{B}}, r_{\mathbf{B}})$ is a tensor category. □

# 7 Tensor Invariance

In this section we present the definition of a monoidal functor. Such a functor preserves properties of the tensor product. In an ideal situation, we would like tensor products to be "strictly associative," that is, the tensor product of $n$ objects should be the equal no matter the bracketing procedure. Then a monoidal functor between categories would be such that we requre the tensor product be preserved. That is, $F(N \otimes M) = F(N) \otimes F(M)$ and $F(f \otimes g) = F(f) \otimes F(g)$ for any objects $N, M$ and morphisms $f, g$. However, this fails for the category of $R$-modules that we have mentioned. We find a compromise by usage of the concept of natural transformations and preservation of identity structure via commutative properties.

We give two s examples of monoidal functors; one from from **Set** to $\mathbf{Mod}(R)$ and another from **Braid** to $\mathbf{Mod}(R)$, for some commutative ring $R$.

## 7.1 Monoidal Functor

Let $(\mathbb{A}, \boxtimes, I, \alpha, l, r)$ and $(\mathbb{B}, \otimes, I', \alpha', l', r')$ be two monoidal categories. We have that $\boxtimes : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\otimes : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$. Suppose $F$ is a functor from $\mathbb{A}$ to $\mathbb{B}$. We recall that we can take products of two functors (see example 45), so that $F \times F : \mathbb{A} \times \mathbb{A} \to \mathbb{B} \times \mathbb{B}$. Then recall that we can take compositions of functors as well (see proposition 52) so that we can compose

$$(F \times F) \circ \otimes = \otimes(F\bullet, F\bullet)$$

which takes to objects $A, B \in Ob(\mathbb{A})$ to $\otimes(F(A), F(A))$, and morphisms $f, g \in \mathrm{Hom}(\mathbb{A})$ to $\otimes(F(f), F(g))$.

We can also compose

$$F \circ \boxtimes$$

which takes $A, B \in Ob(\mathbb{A})$ to $F(A \boxtimes B)$, and morphisms $f, g \in \mathrm{Hom}(\mathbb{A})$ to $F(f \boxtimes g)$.

We note that both functors are from $\mathbb{A} \times \mathbb{A}$ to $\mathbb{B}$.

**Definition 204.** [8, page 50, 51] For monoidal categories $(\mathbb{A}, \boxtimes, I, \alpha, l, r)$ and $(\mathbb{B}, \otimes, I', \alpha', l', r')$, we will say that triple $(F, \phi, \psi)$ is a "monoidal functor" iff:

1. $F$ is a functor from $\mathbb{A}$ to $\mathbb{B}$.

2. $\phi$ is a natural isomorphism from $\otimes(F\bullet, F\bullet) : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ to $F \circ \boxtimes : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$.

   What this means is, given any two objects $A, B$ in $\mathbb{A}$, we have that $\phi_{A,B} : F(A) \otimes F(B) \to F(A \boxtimes B)$ is an isomorphism in $\mathbb{B}$ and given any morphism $(f, g) : (A, B) \to (A', B')$ in $\mathbb{A} \times \mathbb{A}$, the diagram

$$\begin{tikzcd} F(A) \otimes F(B) \arrow[r, "\phi_{A,B}"] \arrow[d, "F(f)\otimes F(g)"'] & F(A \boxtimes B) \arrow[d, "F(f\boxtimes g)"] \\ F(A') \otimes F(B') \arrow[r, "\phi_{A',B'}"'] & F(A' \boxtimes B') \end{tikzcd}$$

   commutes.

3. $\psi$ is an isomophism from $I'$ to $F(I)$ of objects in $\mathbb{B}$.

such that the diagrams

$$\begin{tikzcd} (F(A) \otimes F(B)) \otimes F(C) \arrow[r, "\alpha'_{F(A),F(B),F(C)}"] \arrow[d, "\phi_{A,B}\otimes \mathrm{id}_{F(C)}"'] & F(A) \otimes (F(B) \otimes F(C)) \arrow[d, "\mathrm{id}_{F(A)}\otimes\phi_{B,C}"] \\ F(A \boxtimes B) \otimes F(C) \arrow[d, "\phi_{A\otimes B,C}"'] & F(A) \otimes F(B \boxtimes C) \arrow[d, "\phi_{A,B\otimes C}"] \\ F((A \boxtimes B) \boxtimes C) \arrow[r, "F(\alpha_{A,B,C})"'] & F(A \boxtimes (B \boxtimes C)) \end{tikzcd}$$

$$
\begin{tikzcd}
F(A) \otimes I' \arrow[r, "r'_{F(A)}"] \arrow[d, "\mathrm{id}_{F(A)} \otimes \psi"'] & F(A) \arrow[d, "F(r_A^{-1})"] \\
F(A) \otimes F(I) \arrow[r, "\phi_{A,I}"'] & F(A \boxtimes I)
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes F(A) \arrow[r, "l'_{F(A)}"] \arrow[d, "\psi \otimes \mathrm{id}_{F(A)}"'] & F(A) \arrow[d, "F(l_A^{-1})"'] \\
F(I) \otimes F(A) \arrow[r, "\phi_{I,A}"'] & F(I \boxtimes A)
\end{tikzcd}
$$

commute.

### 7.2 Monoidal Functor from Set to Mod($R$)

We here will describe an example of a monoidal functor from **Set** to **Mod**($R$). To do that, we develop some mechanism first. Recall that **Set** is a monoidal category (see 72). Given set $A$, denote $Free_R(A)$ its free module over ring $R$. Let $\mathbf{Free}_R(A)$ denote the category which has all functions from $A$ to some $R$-module $M$, and let the morphisms between them be $R$-module homomorphisms $\eta$ such that the following diagram commutes for all objects $f, g$ of $\mathbf{Free}_R(A)$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & M \arrow[d, "\eta"] \\
& N
\end{tikzcd}
$$

Then the free module $Free_R(A)$ with its associated inclusion map $inc : A \to Free_R(A)$ is initial in this category. This is the so called universal property of the free module. Given function $f : A \to B$ between sets, we therefore have that there exists unique map $f_R : Free_R(A) \to Free_R(B)$ that restricts to $f$ on $A$.

Then the pair of associations from **Set** to **Mod**($R$) for commutative ring $R$ which takes

$$A \mapsto Free_R(A)$$

$$f \mapsto f_R$$

is a functor, and we denote it as $\mathscr{F}_{\mathbf{Set},\mathbf{Mod}(R)}$. Given any sets $A, B$, and free modules $Free_R(A), Free_R(B)$, define map $g$ from $A \times B$ to $Free_R(A) \otimes Free_R(B)$ as the map which takes

$$g : (a, b) \mapsto a \otimes b$$

Then by the universal property that we have noted, there exists unique $\phi_{A,B}$ making the diagram

$$
\begin{tikzcd}
A \times B \arrow[r, "inc"] \arrow[dr, "g"'] & Free_R(A \times B) \arrow[d, "\phi_{A,B}"] \\
& Free_R(A) \otimes Free_R(B)
\end{tikzcd}
$$

commute. On the other hand, by the universal property of the tensor product, given the bilinear map

$$\xi : \left( \sum_{a \in A} \lambda_a a, \sum_{b \in B} \lambda_b b \right) \mapsto \sum_{a \in A, b \in B} \lambda_a \lambda_b (a, b)$$

there exists $\overline{\xi}$ that makes the diagram

$$\begin{tikzcd} Free_R(A) \times Free_R(B) \arrow[r, "\otimes"] \arrow[dr, "\xi"'] & Free_R(A) \otimes Free_R(B) \arrow[d, "\overline{\xi}"] \\ & Free_R(A \times B) \end{tikzcd}$$

commute. Then $\overline{\xi} \circ \phi_{A,B}$ and $\phi_{A,B} \circ \overline{\xi}$, are identity and it follows that $\phi_{A,B}$ is an isomorphism. It amounts to the more visually appealing correspondence:

$$\left(\sum_{a \in A} \lambda_a a\right) \otimes \left(\sum_{b \in B} \lambda_b b\right) \doteq \sum_{a \in A, b \in B} \lambda_a \lambda_b (a, b)$$

Denote $\phi := \{\phi_{A,B}\}_{A,B \in Ob(\mathbf{Set})}$. Note that the free module generated by the singleton set is isomorphic to $R$. Denote the morphism as $\psi$.

**Example 205.** The triple $(\mathscr{F}_{\mathbf{Set},\mathbf{Mod}(R)}, \phi, \psi)$ as described above is a monoidal functor from **Set** to $\mathbf{Mod}(R)$.

The next subsection gives a detailed exhibition of another monoidal functor.

## 7.3 Example of a Monoidal Functor from Braid to Mod($R$)

We here exhibit an example of a monoidal functor from **Braid** to $\mathbf{Mod}(R)$, where $R$ is commutative. We will also consider a similar example in the case when $R = \mathbb{C}$, the complex numbers.

**Definition 206.** For commutative ring $R$ and $n \in \mathbb{N}$, define $M^{\otimes 0} := R$, $M^{\otimes 1} := M$, and recursively, $M^{\otimes n+1} := M^{\otimes n} \otimes M$.

**Definition 207.** Given $R$-module $M$ and $n \in \mathbb{N}$, where $R$ is commutative, we will denote $\phi_{i,j}$ as the $R$-linear map of $M^{\otimes n}$ which takes for

$$\sum_{\mathbf{x}} \lambda_{\mathbf{x}} x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n \mapsto \sum_{\mathbf{x}} \lambda_{\mathbf{x}} x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n$$

where we denote $\mathbf{x} = x_1 \otimes \cdots \otimes x_n$.

**Proposition 208.** *The $R$-linear map defined by the above association exists and is well defined.*

*Proof.* This is a routine verification and will therefore be skipped. □

Recall that from corollary 194, given any braid class, there exists a braid equal to a composition of underswaps and overswaps that represents the class. Recall that we denote $X_+^i$ as the underswap of $i$ and $i+1$, and $X_-^i$ as the overswap of $i$ and $i+1$. Then for any braid class $[T] \in \mathrm{Hom}(\mathbf{Braid})$ there exists (possibly empty) sequence $(B_1, \cdots, B_n)$ of underswap or overswap braids, such that $[T] = [B_1 \circ \cdots \circ B_n]$, called the "swap decomposition" of $[T]$. The composition of no elements is the identity braid by definition.

**Definition 209.** Given an underswap $X_+^i$, denote the linear map $\Phi(X_+^i) := \phi_{i,i+1}$.

**Definition 210.** Given an underswap $X_-^i$, denote the linear map $\Phi(X_-^i) := \phi_{i,i+1}$.

Suppose $\zeta \in \mathbb{R}$. Then define the following.

**Definition 211.** In particular, when $R = \mathbb{C}$, given an underswap $X_+^i$, denote the linear map $\Psi_\zeta(X_+^i) := e^{\frac{2\pi}{\zeta}} \phi_{i,i+1}$.

**Definition 212.** In particular, when $R = \mathbb{C}$, given an underswap $X_-^i$, denote the linear map $\Psi_\zeta(X_-^i) := e^{-\frac{2\pi}{\zeta}} \phi_{i,i+1}$.

**Definition 213.** Given a braid class $[T]$ and its swap decomposition $(B_1, \cdots, B_n)$, denote

$$\Phi(B_1, \cdots, B_n) := \Phi(B_1) \circ \cdots \circ \Phi(B_n)$$

$$\Psi_\zeta(B_1, \cdots, B_n) := \Psi_\zeta(B_1) \circ \cdots \circ \Psi_\zeta(B_n)$$

When the sequence is empty, such as in the case of the identity braid of type $n$, we define it to be the identity map of $M^{\otimes n}$.

**Definition 214.** Denote $\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}$ as the pair of associations which takes

$$[[n]] \mapsto M^{\otimes n}$$

$$[T] \mapsto \Phi(B_1, \cdots, B_n)$$

where $(B_1, \cdots, B_n)$ is some swap decomposition of $T$.

It requires to show that the association of braids to linear maps is well defined. This is proved below.

*Proof.* Suppose $T \sim T'$ and $(B_1, \cdots, B_n)$ is some swap decomposition of $T$ and $(B'_1, \cdots, B'_m)$ is some swap decomposition of $T'$. We therefore have that

$$B_1 \circ \cdots \circ B_n \sim T \sim T' \sim B'_1 \circ \cdots \circ B'_m$$

and thus by proposition 199, we have that $B_1 \circ \cdots \circ B_n$ and $B'_1 \circ \cdots \circ B'_m$ are related by a sequence of Reidemeister transformations. That is to say there exists sequence of $\rho_1, \cdots, \rho_N$ such that each $\rho_i$ is an operation of the form in definition 196 and in definition 197, and

$$\rho_N \cdots \rho_1(B_1, \cdots, B_n) = (B'_1, \cdots, B'_m)$$

It therefore suffices to show that

$$\Phi(\rho(B_1, \cdots, B_n)) = \Phi(B_1, \cdots, B_n)$$

for any Reidemeister transformation $\rho$ that we have given in definition 196.

We have that the Reidemeister transformation between $2A$ and $2B$ corresponds to the operation

$$(B_1, \cdots, B_j, X_+^i, X_-^i, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, B_{j+3}, \cdots, B_n)$$

and we have that

$$\begin{aligned}(\Phi X_+^i \circ \Phi X_-^i)&(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n)\\ &= \Phi X_+^i(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n)\\ &= (x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n)\end{aligned}$$

So this becomes identity. Therefore Applying $\Phi$, we get

$$\Phi B_1 \circ \cdots \circ \Phi B_j \circ \Phi X_+^i \circ \Phi X_-^i \circ \Phi B_{j+3} \circ \cdots \circ \Phi B_n$$

$$= \Phi B_1 \circ \cdots \circ \Phi B_j \circ \Phi B_{j+3} \circ \cdots \circ \Phi B_n$$

The Reidemeister transformation between $2B$ and $2C$ is similarly cancalled under $\Phi$:

$$\Phi B_1 \circ \cdots \circ \Phi B_j \circ \Phi X_-^i \circ \Phi X_+^i \circ \circ \Phi B_{j+3} \circ \cdots \circ \Phi B_n$$

$$= \Phi B_1 \circ \cdots \circ \Phi B_j \circ \Phi B_{j+3} \circ \cdots \circ \Phi B_n$$

The Reidemeister transformation between $3A$ to $3B$ corresponds to the operation

$$(B_1, \cdots, B_j, X_+^i, X_+^{i+1}, X_+^i, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, X_+^{i+1}, X_+^i, X_+^{i+1}, B_{j+3}, \cdots, B_n)$$

A simple calculation obtains $\Phi X_+^i \circ \Phi X_+^{i+1} \circ \Phi X_+^i = \Phi X_+^{i+1} \circ \Phi X_+^i \circ \Phi X_+^{i+1}$ and therefore the move is cancelled under $\Phi$.

So we conclude that $\Phi(\rho(B_1, \cdots, B_n)) = \Phi(B_1, \cdots, B_n)$ for any Riedemeister transformation $\rho$ that we have mentioned.

We finally need to show that the planar isotopy move where for any $i, j$ such that $i \neq j+1$ or $j \neq i+1$:

$$(B_1, \cdots, B_j, X_\pm^i, X_\pm^j, B_{j+3}, \cdots, B_n) \longleftrightarrow (B_1, \cdots, B_j, X_\pm^i, X_\pm^j B_{j+3}, \cdots, B_n)$$

does not change the result under $\Phi$. Indeed, observe that the following diagram commutes for appropriate $i, j$:

$$\begin{tikzcd}
x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_n \arrow[r, mapsto] \arrow[d, mapsto] & x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \cdots \otimes x_{j+1} \otimes x_j \otimes \cdots \otimes x_n \arrow[d, mapsto] \\
x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_n \arrow[r, mapsto] & x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \cdots \otimes x_{j+1} \otimes x_j \otimes \cdots \otimes x_n
\end{tikzcd}$$

and the result follows. □

**Proposition 215.** *The pair of associations $\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}$ as defined above is a functor.*

*Proof.* We have that $\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}$ maps objects in **Braid** to objects in $\mathbf{Mod}(R)$, and morphisms in **Braid** to morphisms in $\mathbf{Mod}(R)$.

$$[[n]] \mapsto M^{\otimes n}$$

$$[T] \mapsto \Phi(B_1, \cdots, B_n)$$

By definition we have that if $[T]$ is of type $n$, then $\Phi(B_1, \cdots, B_n)$ is a linear map from $M^{\otimes n}$ to $M^{\otimes n}$.

We show that composition and identity is respected.

Suppose $[T], [T']$ are two composible braids; then take swap decomposition $(B_1, \cdots, B_n)$ of $T$, and swap decomposition $(B'_1, \cdots, B'_m)$ of $T'$. Then

$$[T'] \circ [T]$$

$$= [T' \circ T]$$

$$= [B'_1 \circ \cdots \circ B'_m \circ B_1 \circ \cdots \circ B_n]$$

and so

$$\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}([T'] \circ [T])$$

$$= \Phi(B'_1 \circ \cdots \circ B'_m \circ B_1 \circ \cdots \circ B_n)$$

$$= \Phi B'_1 \circ \cdots \circ \Phi B'_m \circ \Phi B_1 \circ \cdots \circ \Phi B_n$$

$$= \Phi(B'_1 \circ \cdots \circ B'_m) \circ \Phi(B_1 \circ \cdots \circ B_n)$$

$$= \mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}([T']) \circ F([T])$$

Suppose $\mathrm{id}_n$ is an identity braid from $[[n]]$ to $[[n]]$. Then by definition it is the identity map of $M^{\otimes n}$. So $\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}$ preserves identity. □

**Definition 216.** Denote $\mathscr{G}_{\mathbf{Braid},\mathbf{Mod}(R)}$ as the pair of associations which takes

$$[[n]] \mapsto M^{\otimes n}$$

$$[T] \mapsto \Psi_\zeta(B_1, \cdots, B_n)$$

where $(B_1, \cdots, B_n)$ is some swap decomposition of $T$.

It can also be proven that $\mathscr{G}_{\mathbf{Braid},\mathbf{Mod}(R)}$ is well defined and is a functor in a very similar manner. We will not show it here.

**Definition 217.** Denote $\phi_{\mathbf{Braid},\mathbf{Mod}(R)}([[n]], [[m]])$, or more succinctly $\phi(n, m)$, as the canonical isomorphism from $M^{\otimes n} \otimes M^{\otimes m}$ to $M^{\otimes n+m}$ (see corollary 84 and corollary 86).

**Definition 218.** Define $\psi_{\mathbf{Braid},\mathbf{Mod}(R)}$ as the identity map from $R$ to $R$.

**Proposition 219.** *The triple* $(\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}, \phi_{\mathbf{Braid},\mathbf{Mod}(R)}, \psi_{\mathbf{Braid},\mathbf{Mod}(R)})$ *as defined above is a monoidal functor from* $(\mathbf{Braid}, \boxtimes_{\mathbf{B}}, I_{\mathbf{B}}, \alpha_{\mathbf{B}}, l_{\mathbf{B}}, r_{\mathbf{B}})$ *to* $(\mathbf{Mod}(R), \otimes, R, \alpha, l, r)$.

*Proof.* We shall go through the conditions. For brevity, we shorten $\mathscr{F}_{\mathbf{Braid},\mathbf{Mod}(R)}$, $\phi_{\mathbf{Braid},\mathbf{Mod}(R)}$, $\psi_{\mathbf{Braid},\mathbf{Mod}(R)}$ to $\mathscr{F}, \phi, \psi$.

1. $\mathscr{F}$ is a functor from **Braid** to $\mathbf{Mod}(R)$.

2. We want to show that given any two objects $[[n]], [[m]]$ in **Braid**, and given any morphism $(T, U) : ([[n]], [[m]]) \to ([[n]], [[m]])$ in **Braid** $\times$ **Braid**, the diagram

$$\begin{tikzcd} M^{\otimes n} \otimes M^{\otimes m} \arrow[r, "\phi(n,m)"] \arrow[d, "\mathscr{F}(T)\otimes\mathscr{F}(U)"'] & M^{\otimes n+m} \arrow[d, "\mathscr{F}(T\boxtimes U)"] \\ M^{\otimes n} \otimes M^{\otimes m} \arrow[r, "\phi(n,m)"'] & M^{\otimes n+m} \end{tikzcd}$$

commutes. The tensor product of the two $R$-linear maps then is one such that $\mathscr{F}(T)$ is applied to the first $n$ tensor products and $\mathscr{F}(U)$ is applied to the last $m$ tensor products. Now we have that since $T$ is a braid on only the first $n$ points, we have that $\mathscr{F}(T)$ only affects $\sum \lambda_{\mathbf{x}} \otimes x_1 \otimes \cdots \otimes x_n$ and $U$ is a braid on the next $m$ points, $\mathscr{F}(U)$ only affects $\sum \lambda_{\mathbf{y}} y_1 \otimes \cdots \otimes y_n$. So the diagram shown in appendix A commutes. In short, one sees that whether we switch around two sequences $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_m)$ individually and then concatenate them is the same as concatenating them and then switching each of them individually; for permutations $\kappa, \gamma$, we illustrate in the following diagram:

$$\begin{tikzcd} (x_1, \cdots, x_n), (y_1, \cdots, y_m) \arrow[r, mapsto] \arrow[d, mapsto] & (x_1, \cdots, x_n, y_1, \cdots, y_m) \arrow[d, mapsto] \\ (x_{\gamma(1)}, \cdots, x_{\gamma(n)}), (y_{\kappa(1)}, \cdots, y_{\kappa(m)}) \arrow[r, mapsto] & (x_{\gamma(1)}, \cdots, x_{\gamma(n)}, y_{\kappa(1)}, \cdots, y_{\kappa(m)}) \end{tikzcd}$$

So $\phi$ is a natural isomorphism from $\otimes(\mathscr{F}\bullet, \mathscr{F}\bullet) : \mathbf{Braid} \times \mathbf{Braid} \to \mathbf{Mod}(R)$ to $\mathscr{F} \circ \boxtimes : \mathbf{Braid} \times \mathbf{Braid} \to \mathbf{Mod}(R)$ as desired.

3. $\psi$ is an isomophism from $R$ to $\mathscr{F}(I_{\mathbf{B}}) = \mathscr{F}(\varnothing) = R$ of objects in $\mathbf{Mod}(R)$. For any $[[n]], [[m]], [[u]]$ of **Braid**, we have that the diagram

$$\begin{tikzcd} (M^{\otimes n} \otimes M^{\otimes m}) \otimes M^{\otimes u} \arrow[r, "\alpha(n,m,u)"] \arrow[d, "\phi(n,m)\otimes\mathrm{id}"'] & M^{\otimes n} \otimes (M^{\otimes m} \otimes M^{\otimes u}) \arrow[d, "\mathrm{id}\otimes\phi(m,u)"] \\ M^{\otimes n+m} \otimes M^{\otimes u} \arrow[d, "\phi(n+m,u)"'] & M^{\otimes n} \otimes M^{\otimes m+u} \arrow[d, "\phi(n,m+u)"] \\ M^{\otimes n+m+u} \arrow[r, "\mathrm{id}"'] & M^{\otimes n+m+u} \end{tikzcd}$$

commutes, by confirming that the two paths of evaluation coincide for any element in $(M^{\otimes n} \otimes M^{\otimes m}) \otimes M^{\otimes u}$, and the diagram

$$\begin{tikzcd} M^{\otimes n} \otimes R \arrow[r, "r'_{M^{\otimes n}}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"'] & M^{\otimes n} \arrow[d, "\mathrm{id}"] \\ M^{\otimes n} \otimes R \arrow[r, "\phi(n,0)"'] & M^{\otimes n+0} \end{tikzcd} \qquad \begin{tikzcd} R \otimes M^{\otimes n} \arrow[r, "l'_{M^{\otimes n}}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"'] & M^{\otimes n} \arrow[d, "\mathrm{id}"] \\ R \otimes M^{\otimes n} \arrow[r, "\phi(0,n)"'] & M^{\otimes n+0} \end{tikzcd}$$

commutes by definition of $l', r'$, and $\phi$.

□

**Proposition 220.** *$(\mathscr{G}_{\mathbf{Braid},\mathbf{Mod}(R)}, \phi_{\mathbf{Braid},\mathbf{Mod}(R)}, \psi_{\mathbf{Braid},\mathbf{Mod}(R)})$ as defined above is a monoidal functor from $(\mathbf{Braid}, \boxtimes_{\mathbf{B}}, I_{\mathbf{B}}, \alpha_{\mathbf{B}}, l_{\mathbf{B}}, r_{\mathbf{B}})$ to $(\mathbf{Mod}(R), \otimes, R, \alpha, l, r)$.*

*Proof.* Exactly the same as in the proof of the previous proposition, except for adding in the phase terms when proving condition 2. □

# 8 Further Topics

In this section we give an overview of futher topics relating to categorical quantum mechanics and topological quantum computing which we could not go into detail due to length and time constraints.

### The Yang-Baxter Equation

[12, page 167] Given vector space $V$ over field $k$, a morphism $\check{R}$ of $V \otimes V$ is said to satisfy the the Yang-Baxter equation iff it satisfies

$$(\check{R} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \check{R})(\check{R} \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes \check{R})(\check{R} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \check{R})$$

The Yang-Baxter equation appears in the field of integrable systems in quantum mechanics. Reidemeister move 3 corresponds to this equation.

### Category of Hilbert Spaces

The set of all Hilbert spaces forms a submonoidal category of $\mathbf{Mod}(\mathbb{C})$. It in turn contains the submonoidal category of finite Hilbert spaces. The state of a quantum system is assumed to be a non-zero element in some Hilbert space. Any finite dimensional Hilbert space is isomorphic to $\mathbb{C}^n$ for dimension $n$, thus any finite dimensional quantum system is represented by some vector in $\mathbb{C}^n$ space.

An element of a two dimensional Hilbert space $\mathscr{H}$ is said to be a qubit. It can be represented as $(a, b) \in \mathbb{C}^2$. The simulation of $n$ qubits is simply the tensor product $\mathscr{H}^{\otimes n}$, which has dimension $2^n$.

### Anyons

We introduce notation given by [18, page 56, 57]. We will denote 1 as the vacuum state. Given particle $a$, we denote $\overline{a}$ as its antiparticle. It may be the case that a particle is its own antiparticle. From a vacuum, a particle and antiparticle occur in pairs.

[18, page 57] Given a finite set of particles $1, a, b, c, \ldots$, we have compact notation known as "fusion rules". Given two particles $a$ and $b$, we denote their fusion as $a \times b$. We have that $a \times b = b \times a$. We denote $a \times b = N^c_{a,b} c + N^d_{a,b} d + \cdots$ to mean that fusing together $a$ and $b$ obtains $c$ or particle $d$, and so on for all particles in the sum, where $N^c_{a,b}$ is the number which tells us how many different ways there are to obtain particle $c$. The method of preparation of particles and process of fusion uniquely determines the fusion outcome. "Method" here means braiding and the order with which we fuse the particles. Given any set of fusion rules, it is possible to treat particles as proper variables and algbraically obtain new equations from old ones. Abelian anyons and Ising anyons are examples which clarify this.

We initialize the quantum state by creating some anyons in some order. The application of quantum gates corresponds to braiding these anyons. The final state, which is the outcome of the computation, is then measured. [18, page 66]. Therefore the observable has the eigenstates given by

$$|(a, b) \to c, \mu\rangle$$

where $c$ is the fusion outcome of fusing $a$ and $b$, and $\mu$ is the number of degenerate states; ranging from 1 to $N_{a,b}^c$ [18, page 59].

### Abelian Anyons

Abelian anyons are those which satisfy the fusion rule $a \times b = c$ for all particles $a, b, c$. [18, page 57]. Therefore, given $n$ particles, it does not matter how we fuse them; the fusion outcome is the same.

### Ising Anyons

[13, page 17] The Ising anyon model has particles $1, \psi, \sigma$, where $\psi$ is a fermion and $\sigma$ is an anyon. They satisfy the fusion rules:

$1 \times 1 = 1$, $1 \times \psi = \psi$, $1 \times \sigma = \sigma$, $\psi \times \psi = 1$, $\psi \times \sigma = \sigma$, $\sigma \times \sigma = 1 + \psi$.

We have that

$$\sigma \times \sigma \times \sigma = \sigma \times (1 + \psi) = \sigma + \sigma \times \psi = \sigma + \sigma = 2\sigma$$

So we know that there are two ways to obtain the fusion result $\sigma$. The first is by combining $\sigma$ with the vacuum state that resulted from $\sigma \times \sigma$ and the second is by combining $\sigma$ with $\psi$, which was created from the alternative method of combining $\sigma$ with $\sigma$. Braiding also determines whether we get 1 or $\psi$ when combining $\sigma$ with $\sigma$ as seen in [18, page 69].

Although Ising anyons are the most physically viable method [18, page 3][13, page 13], they cannot perform all desired logical gates required for computation due to their braiding properties and therefore cannot realize a universal computational device [13, page 18, 19]. On the contrary, Fibonacci anyons do allow all logical gates and therefore are a candidate for realizing a quantum computer. [13, page 13, 17].

### Graphical Notation of Categories and Graphical Calculus

Graphical calculus is a more visual notation for category theory, and a powerful one to obtain immediate results when dealing with monoidal categories. An explanation of this calculus can be found in [8, page 15, 16]. A further explanation of graphical calculus for monoidal categories is found in [8, page 39-41]. A crucial "correctness" theorem states that two morphisms with their respective graphs are equal iff the graphs are planar isotopic [8, page 39-40].

Joyal and Street provided a rigorous formulation of the method of graphical calculus for monoidal categories [10] and braided monoidal categories [11].

### Braided Monoidal Categories

A braided monoidal category is a monoidal category with an additional natural isomorphism structure that satisfies the hexagon axioms as given in [8, page 45, 46]. Invertibility of the natural isomorphism is then graphically the same as Reidemeister move 2. A similar correctness theorem is true for the graphical calculus of braided monoidal categories [8, page 47].

### Strictification and The Coherence Theorem of Mac Lane

A monoidal category $(\mathbb{A}, \otimes, I, \alpha, l, r)$ is said to be strict iff $\alpha, l, r$ are identity morphisms between functors. The category of tangles is an example of a strict category while the category of $R$-modules is a counterexample.

A functor is said to be an "equivalence of categories" iff it is fully faithful and essentially surjective. Then a monoidal equivalence is a monoidal functor which is fully faithful and essentially surjective [8, page 52]. Two monoidal categories are said to be "monoidally equivalent" if there exists a monoidal equivalence between them. The strictification theorem (a.k.a. Mac Lane Strictness theorem) states that for any monoidal category, there exists a strict monoidal category that is equivalent to it [8, page 53]. This result can be used to prove the Mac Lane coherence theorem as given in [5, page 40].

### Dagger Categories

An involutive functor $F$ is one which when applied twice to any morphism $f$, satisfies $F(F(f)) = f$. Noting that the dagger operation on a Hilbert spaces is a contravariant functor that takes morphisms to their adjoints, it motivates the general definition that a contravariant functor $F : \mathbb{A} \to \mathbb{A}$ is called a "dagger" iff it is identity on objects and is involutive [8, page 75].

The category **Fib** of Fibonacci anyons is an example of a braided monoidal dagger category [8, page 80].

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# Appendices

## A Commutate Diagram

$$
\begin{tikzcd}
\left(\sum \lambda_x x_1 \otimes \cdots \otimes x_n\right) \otimes \left(\sum \lambda_y y_1 \otimes \cdots \otimes y_m\right) \arrow[r, "\phi_{[[n]],[[m]]}"] \arrow[d, "\mathscr{F}(T) \otimes \mathscr{F}(U)"'] & \sum \lambda_x \lambda_y x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m \arrow[d, "\mathscr{F}(T \boxtimes U)"] \\
\left(\mathscr{F}(T) \sum \lambda_x x_1 \otimes \cdots \otimes x_n\right) \otimes \mathscr{F}(U) \left(\sum \lambda_y y_1 \otimes \cdots \otimes y_m\right) \arrow[r, "\phi_{[[n]],[[m]]}"] & \mathscr{F}(T \boxtimes U) \sum \lambda_x \lambda_y x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m
\end{tikzcd}
$$